Non-intersecting directed polymers

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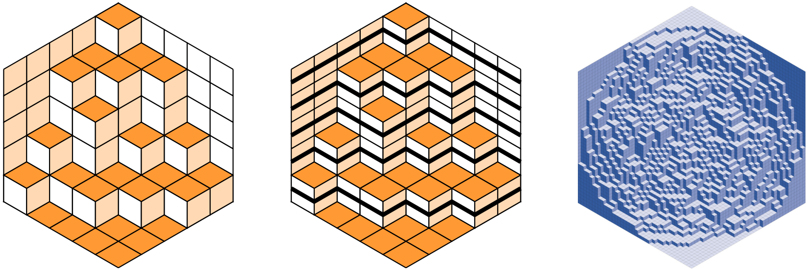
Coulomb gases and universality, Jussieu

Non-intersecting paths and random surfaces

The configurations in various statistical mechanics models can be encoded by non-intersecting paths.



- Height function

$$h(x,y) = \text{ number of paths below the point } (x,y)$$

Non-intersecting paths and the GFF

One of the most well-known models of non-intersecting paths is Dyson's Brownian motion [Dyson 1962]: eigenvalues $\lambda_1(t) < \cdots < \lambda_n(t)$ of a Hermitian random matrix with Brownian entries

$$d\lambda_i(t) = dB_i(t) + \sum_{j \neq i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}$$

The $\lambda_i(t)$ can be seen as Brownian motions conditioned never to intersect.

- The height function of Dyson Brownian motion on the circle converges to the Gaussian free field [Spohn 1998]. Many extensions and variants exist (cf Gaultier Lambert's talk!);
- Non-intersecting paths from GUE corners process converge to the GFF [Borodin 2010], as well as discrete analogues related to Schur measures [Borodin-Ferrari 2008] and anisotropic KPZ models;
- Height function of lozenge tilings, dimer models [Kenyon 2001], universality for lozenge tilings conjectured in [Kenyon-Okounkov 2003]; more general discrete non-intersecting walks [Gorin-Petrov 2016];
- In Physics, this universality class has been studied in many papers starting with [de Gennes, 1968].

Log-correlated models

At a fixed time, the distribution of Dyson's Brownian motion is the GUE density, that is a $1d$ log-gas

$$\prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_i e^{\frac{-\lambda_i^2}{2t}}$$

- This is a determinantal point process. In the bulk, the microscopic behaviour is described by the determinantal point process with sine kernel
$$K(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)}.$$
- This defines log-correlated fields with universal properties. In particular, the variance of the height (counting) function in the bulk is of order $\log(n)$.

Non-intersecting paths with disorder

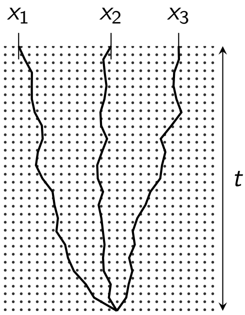
Consider non-intersecting paths in a disordered environment (random walks or diffusions in random environment, directed polymer models) [Kardar 1987, Emig-Kardar 2000, ...].

What are the analogues of

- The Gaussian free field ?
- The Sine process ?
- log correlations ?
  ⇝ Open problem. Some related models exhibit log squared correlations at small enough temperature [Toner-DiVicenzo 1990].
- The 1d log-gas ?
  ⇝ The partition function

$$Z(x_1, \ldots, x_n; t)$$

of n non-intersecting Brownian directed polymers in a white noise environment.



Outline

1. Continuous directed polymers

2. Stationary measures of non-intersecting directed polymers

3. Ideas of proofs through an integrable discrete model, the log-gamma polymer

Continuous directed polymers

- The continuous directed polymer model is a probability measure on continuous paths $t \mapsto W_t$ proportional to

$$\exp\left(\int_0^t ds \xi(W_s, s)\right) \mathcal{W}(W),$$

where $\mathcal{W}$ is the Brownian measure, and $\xi(x,t)$ is a space-time white noise.

- Define the partition function of a single polymer paths as

$$Z(x,s|y,t) = p_{t-s}(x,y)\mathbf{E}_{\mathcal{W}}\left[e^{\int_s^t du \xi(W_u,u)}\right]$$

where W is a Brownian bridge from x to y, p_t is the heat kernel, and $\mathbf{E}_{\mathcal{W}}$ is the expectation w.r.t. to the Brownian bridge measure.

- We define the (quenched) endpoint measure

$$\mathcal{P}(x) = \frac{Z(0,0|x,t)}{\int_{\mathbb{R}} dy Z(0,0|y,t)}.$$

KPZ equation and universality class

The partition function $Z(0,0|x,t)$ satisfies the multiplicative noise stochastic heat equation

$$\partial_t Z(x,t) = \frac{1}{2}\partial_{xx} Z(x,t) + \xi Z(x,t),$$

with delta function initial condition.
The function $h(x,t) = \log Z(x,t)$ solves the Kardar-Parisi-Zhang equation

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\left(\partial_x h\right)^2 + \xi,$$

This is one representative of a large universality class of models of interface growth described by a height function so that

- $\mathrm{Var}\left[h(x,t)\right]$ grows as $t^{2/3}$ as $t \to \infty$;
- The limiting distribution of $\frac{h(x,t)-ct}{t^{1/3}}$ depends on the initial condition, and is often related to random matrix eigenvalue statistics;
- $\mathrm{Cov}(h(x,t), h(y,t))$ is non trivial for $|x-y| \propto t^{2/3}$.
- The large scale space-time fluctuations are described by a (conjecturally) universal process called the KPZ fixed point [Matetski-Quastel-Remenik 2017] [Dauvergne-Ortmann-Virag 2018].

Non-intersecting polymers

Now we define the partition function of n non-intersecting polymers from points

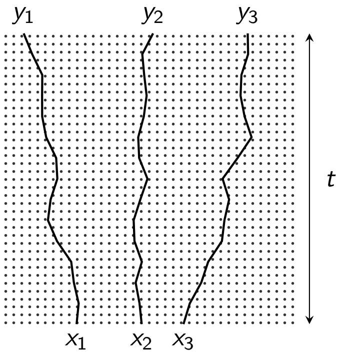
$$x_1 < x_2 < \cdots < x_n$$

at time 0 to points

$$y_1 < y_2 < \cdots < y_n$$

at time t, by the [Karlin-McGregor 1959] determinant

$$Z_n(\vec{x}, 0|\vec{y}, t) = \det\left(Z(x_i, 0|y_j, t)\right)_{i,j=1}^{n}$$



For smooth noise ξ, it satisfies the Feynman Kac representation

$$Z_n(\vec{x}, 0|\vec{y}, t) = \det\left(p_t(x_i, y_j)\right)_{i,j=1}^{\ell} \mathbf{E}\left[e^{-\int_0^t d\tau \sum_j \xi(W_j(\tau), \tau)}\right],$$

where the W_j are non intersecting Brownian bridges.

O'Connell-Warren multilayer SHE

The partition function $Z_n(\vec{x}, 0|\vec{y}, t)$ satisfies (formally) the stochastic PDE

$$\partial_t Z_n = \sum_{i=1}^{n} \frac{1}{2} \partial_{y_i y_i} Z_n + Z_n \sum_{i=1}^{n} \xi(y_i, t)$$

on the Weyl chamber

$$\mathbb{W}_n = \{\vec{y} \in \mathbb{R}^n; y_1 < y_2 < \cdots < y_n\},$$

with the boundary condition that $Z_n = 0$ whenever any $y_j = y_{j+1}$. For each n, this is a Markov process on $C(\mathbb{W}_n, \mathbb{R})$.
[O'Connell-Warren 2011] noticed that defining,

$$\begin{aligned} M_n(x, 0|y, t) &:= \lim_{\substack{\vec{x} \to x \\ \vec{y} \to y}} \frac{Z_n(\vec{x}, 0|\vec{y}, t)}{\prod_{i<j}(x_i - x_j)(y_i - y_j)} \\ &= c_{n,t} \det\left(\partial_x^i \partial_y^j Z(x, 0|y, t)\right)_{i,j=1}^{n}, \end{aligned}$$

the family of processes $(M(x, 0|y, t))_{1 \leqslant i \leqslant n, y \in \mathbb{R}}$ is also a Markov process on $C(\mathbb{R}, \mathbb{R}^n)$, that satisfies a hierarchy of stochastic PDEs, now called the O'Connell-Warren multilayer stochastic heat equation.

Asymptotics

It is known that

$$\frac{\log Z_1(0,0|0,2t)+t/12}{t^{1/3}} \xrightarrow[t\to\infty]{} \lambda_1,$$

where the random variable λ_1 follows the Tracy-Widom GUE distribution function [Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Sasamoto-Spohn, Dotsenko, $\approx$ 2011].

Conjecture

For each $n \geqslant 1$,

$$\frac{\log M_n(0,0|0,2t)+nt/12}{t^{1/3}} \xrightarrow[t\to\infty]{} \lambda_1+\cdots+\lambda_n,$$

where the λ_i have the distribution of the n first eigenvalues of the GUE in the large scale limit at the edge of the spectrum.

Tail bounds are derived in [De Luca-Le Doussal 2017]. The conjecture is expected to hold universally for any $(1+1)-$dim polymer model under some moments assumptions on the noise (proved for some zero temperature models [Borodin-Okounkov-Olshanski 1999]).

Stationary measure and long polymers

Let us go back to a single polymer model. To understand the behaviour of the endpoint measure

$$\mathcal{P}(x) = \frac{Z_1(0,0|x,t)}{\int_{\mathbb{R}} dy Z_1(0,0|y,t)},$$

for large t, it is enough to know that

$$\lim_{t\to\infty} \frac{Z_1(0,0|y,t)}{Z_1(0,0|z,t)} \overset{(d)}{=} \frac{e^{B_1(y)}}{e^{B_1(z)}},$$

where B_1 is a standard Brownian motion [Das-Zhu 2022].

This comes from the **non-trivial** fact that the Brownian motion is a stationary measure for the KPZ equation [Bertini-Giacomin 1997]: if a solution of

$$\partial_t Z(x,t) = \frac{1}{2}\partial_{xx} Z(x,t) + \xi Z(x,t),$$

is such that $Z(x,t=0) = e^{B_1(x)}$, then for all $t > 0$,

$$\frac{Z(x,t)}{Z(z,t)} \overset{(d)}{=} \frac{e^{B_1(x)}}{e^{B_1(z)}}.$$

Question: What is the analogue for n non-intersecting polymers ?

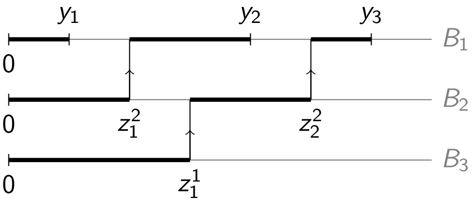
Stationary measure for non-intersecting polymers

[B.-Le Doussal 2022] For any fixed $\vec{x}$,

$$\frac{Z_n(\vec{x},0|\vec{y},t)}{Z_n(\vec{x},0|\vec{z},t)} \xRightarrow[t\to\infty]{} \frac{Z_n^{\mathrm{stat}}(\vec{y})}{Z_n^{\mathrm{stat}}(\vec{z})}$$

where $Z_1^{\mathrm{stat}}(y) = e^{B_1(y)}$, and more generally, $Z_n^{\mathrm{stat}}(\vec{y})$ is defined as a partition function

$$Z_3^{\mathrm{stat}}(\vec{y}) = \int \prod_{i,j} dz_i^j \exp \left(\quad \right)$$



where the z_i^j interlace, i.e. $z_i^{k+1} \leq z_i^k \leq z_{i+1}^{k+1}$, and the weight is the sum over all thick segments of a Brownian increment, i.e.

$$\sum_{i,j} B_j(z_i^j) - B_j(z_{i-1}^{j-1}).$$

- If we average over the noise,

$$\mathbb{E}\left[Z_n^{\mathrm{stat}}(\vec{y})\right] = \prod_{i<j} |y_i - y_j| \prod_{i=1}^{n} e^{y_i/2}.$$

  We recover the 1d log gas.
- In the small-scale limit, $y_i = \epsilon \tilde{y}_i$,

$$Z_n^{\mathrm{stat}}(\vec{y}) \propto \prod_{i<j} |\tilde{y}_i - \tilde{y}_j|.$$

- In the large-scale or zero temperature limit, i.e. $\xi \to \beta\xi$, we obtain a similar result with

$$\log \int \prod_{i,j} dz_i^j \exp\left(\dots\right) \to \sup_{z_i^j}\left(\dots\right).$$

Generalization

If $x_i = -a_i t$ for all $1 \leqslant i \leqslant n$,

$$\frac{Z_n(\vec{x},0|\vec{y},t)}{Z_n(\vec{x},0|\vec{z},t)} \xrightarrow[t\to\infty]{} \frac{Z_n^{\mathrm{stat}}(\vec{y};\vec{a})}{Z_n^{\mathrm{stat}}(\vec{z};\vec{a})}$$

where now, the Brownian motions $B_i(t)$ are replaced by $B_i(t) - a_i t$.

Case $n = 2$, explicit computations

If we condition over the value of the first polymer endpoint y_1, and assume that the drifts $a_1 = a_2 = -a < -1/2$, the endpoint measure becomes normalizable

$$\mathcal{P}(y_1, y_2) = \frac{Z_2^{\mathrm{stat}}(y_1, y_2)}{\int_{y_1}^{+\infty} Z^{\mathrm{stat}}(y_1, y_2) dy_2}.$$

We can compute the cumulants of the difference between the two endpoints [B.-Le Doussal 2022]

$$\mathbb{E}\left[\kappa_k(y_2 - y_1)\right] = (-2)^k (2^k \psi_k(4a) - 3\psi_k(2a)),$$

where κ_k denotes the kth cumulant of the measure $\mathcal{P}$ and the function $\psi_k(z) = \partial_z^k \log \Gamma(z)$ is the polygamma function (it uses results of [Fitzgerald-Warren 2020]).

Open problem Analyze the endpoint measure $\mathcal{P}(y_1, \ldots, y_n)$, and the associated height function as n goes to infinity.

Proof ideas

1. $Z_n^{\mathrm{stat}}(\vec{y})$ is the stationary measure of the stochastic PDE

$$\partial_t Z_n(\vec{y}, t) = \frac{1}{2}\Delta Z_n(\vec{y}, t) + Z_n(\vec{y}, t) \sum_{i=1}^{n} \xi(y_i, t),$$

with Dirichlet boundary condition on $\partial \mathbb{W}_n$, in the sense that if $Z_n(\vec{y}, t = 0) = Z^{\mathrm{stat}}(\vec{y})$, for all t,

$$\frac{Z_n(\vec{y}, t)}{Z_n(\vec{z}, t)} \overset{(d)}{=} \frac{Z_n^{\mathrm{stat}}(\vec{y})}{Z_n^{\mathrm{stat}}(\vec{z})}.$$

For $n = 1$ this is not obvious, though well-known [Bertini-Giacomin 1997]

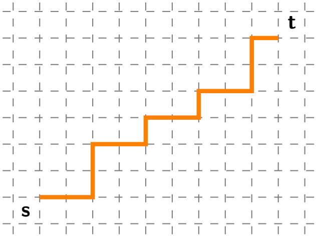
2. We show that a discrete analogue of $Z^{\mathrm{stat}}(\vec{y})$ is a stationary measure for a discrete variant model that is integrable, the log-gamma polymer.

3. The stationary process can be determined either using results on the geometric RSK correspondance [Corwin-O'Connell-Seppäläinen, O'Connell-Warren 2011] or using a general argument based on the symmetries of the model [B.-Corwin 2022].

Log-gamma directed polymer

The model was introduced by [Seppäläinen (2012)]. Let weights $w_{i,j}$ be i.i.d. inverse Gamma random variables with parameter θ, i.e. with density

$$\frac{\mathbb{1}_{w\geqslant 0}}{\Gamma(\theta)} w^{-\theta-1} e^{-1/w}.$$



For $\mathbf{s},\mathbf{t}\in\mathbb{Z}^2$, define the partition function

$$\mathcal{Z}(\mathbf{s}|\mathbf{t}) = \sum_{\pi:\mathbf{s}\to\mathbf{t}} \prod_{(i,j)\in\pi} w_{i,j},$$

where the sum is over upright paths from $\mathbf{s}$ to $\mathbf{t}$.
Similarly, for n-tuples of points $\mathbf{s}_1,\dots,\mathbf{s}_n$ and $\mathbf{t}_1,\dots,\mathbf{t}_n$, we define

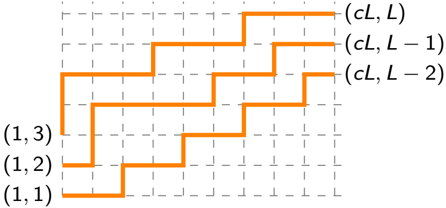
$$\mathcal{Z}_n(\mathbf{s}_1,\dots,\mathbf{s}_n|\mathbf{t}_1,\dots,\mathbf{t}_n) = \sum_{\text{non-intersecting paths}} \prod w_{ij} = \det\left(\mathcal{Z}(\mathbf{s}_i|\mathbf{t}_j)\right)_{i,j=1}^n$$

the partition function for n non intersecting paths.

As the polymer length $L = \|\mathbf{t} - \mathbf{s}\|_1 \to \infty$,

$$\frac{\log \mathcal{Z}(\mathbf{s}|\mathbf{t}) - c_1 L}{c_2 L^{1/3}} \xRightarrow[L\to\infty]{} \lambda_1,$$

where λ_1 has the Tracy-Widom dist. [Borodin-Corwin-Remenik 2012, Krishnan-Quastel 2016, B.-Corwin-Dimitrov 2020]. Many other results exist about spatial correlations, properties of geodesics, etc.



For $\mathbf{s}_i = (1, i)$ and $\mathbf{t}_i = (cL, L - n + i)$, it is conjectured that

$$\frac{\log \mathcal{Z}_n(\mathbf{s}_1, \ldots, \mathbf{s}_n|\mathbf{t}_1, \ldots, \mathbf{t}_n) - nc_1 L}{c_2 L^{1/3}} \xRightarrow[L\to\infty]{} \lambda_1 + \cdots + \lambda_n.$$

[Johnston-O'Connell 2019] conjectured a law of large numbers as the number of non-intersecting polymers grow, i.e. for $n = \alpha L$,

$$\lim_{L\to\infty} \frac{1}{L^2} \log \mathcal{Z}_n(\mathbf{s}_1, \ldots, \mathbf{s}_n|\mathbf{t}_1, \ldots, \mathbf{t}_n) = F(c, \alpha),$$

discrete polymers $\to$ continuous polymers

The partition function satisfies the recurrence,

$$\mathcal{Z}(\mathbf{0}|n,m) = w_{n,m}\left(\mathcal{Z}(\mathbf{0}|n-1,m) + \mathcal{Z}(\mathbf{0}|n,m-1)\right).$$

This is a discrete analogue of the stochastic PDE

$$\partial_t Z = \frac{1}{2}\partial_{xx} Z + \xi Z.$$

and scaling $\theta, n, m \to \infty$ appropriately,

$$\mathcal{Z}(\mathbf{0}|n,m) \xrightarrow[n,m,\theta\to\infty]{} Z(x,t)$$

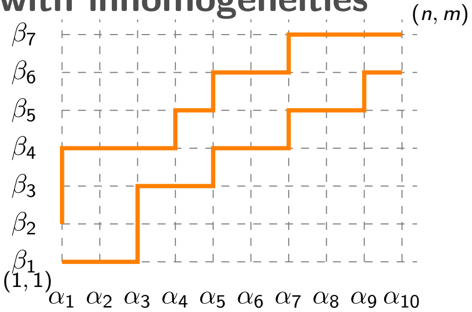
(this is a general result of convergence of discrete polymers at high temperature [Alberts-Khanin-Quastel 2010]).

Partition functions for non-intersecting polymers converge as well thanks to the Karlin-McGregor theorem.

Log-gamma polymer with inhomogeneities

Let $\alpha_1, \alpha_2, \dots,$ and $\beta_1, \beta_2, \dots$ be positive real numbers and take

$w_{i,j} \sim \mathrm{Gamma}^{-1}(\alpha_i + \beta_j)$.



[Corwin-O'Connell-Seppäläinen-Zygouras 2011] proved that

$$\Big(\mathcal{Z}_i\big((1,1)\dots(1,i)|(n,m-i+1),\dots,(n,m)\big)\Big)_{1\leqslant i\leqslant n}$$

has the same distribution as the random vector $(x_1, \dots, x_n)$ with density

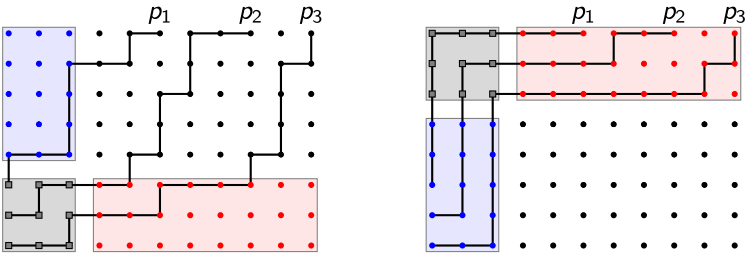
$$\frac{1}{C(\alpha,\beta)}\psi_{\alpha_1,\dots,\alpha_n}(\mathbf{x})\tilde{\psi}_{\beta_1,\dots,\beta_m}(\mathbf{x})d\mathbf{x}$$

where $\psi_{\alpha_1,\dots,\alpha_n}(\mathbf{x})$ and $\tilde{\psi}_{\beta_1,\dots,\beta_m}(\mathbf{x})$ are Whittaker functions. They are **invariant under permutations** of the α_i or the β_i.

Symmetry argument

A stationary model is obtained by letting

$$\begin{cases} \alpha_i = \beta_i = 0 & \text{for } 1 \leqslant i \leqslant n \\ \alpha_i = \alpha & \text{for } i > n \\ \beta_i = \beta & \text{for } i > n \end{cases}$$



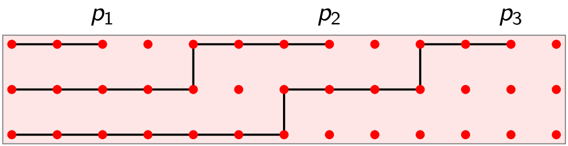
$$\text{weight}(\bullet) \sim \text{Gamma}^{-1}(\alpha+\beta), \quad \text{weight}(\blacksquare) \sim \text{Gamma}^{-1}(0) = +\infty$$

$$\text{weight}(\color{blue}\bullet) \sim \text{Gamma}^{-1}(\beta), \quad \text{weight}(\color{red}\bullet) \sim \text{Gamma}^{-1}(\alpha)$$

Using the symmetry w.r.t. to inhomogeneity parameters, one can exchange rows, so that the partition functions on both sides have the same distribution.

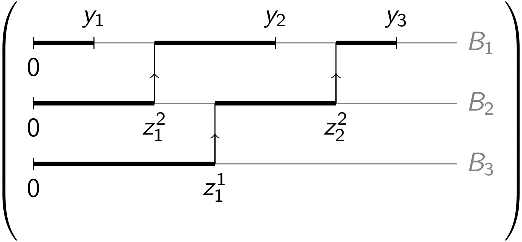
Discrete stationary process $\to Z^{\mathrm{stat}}$

This leads to a discrete stationary process $\mathcal{Z}_3^{\mathrm{stat}}(p_1, p_2, p_3)$ defined as the partition function of



which, under appropriate scaling ($\alpha, \beta \to \infty$, $p_i \to \infty$) becomes

$$Z_3^{\mathrm{stat}}(\vec{y}) = \int \prod_{i,j} dz_i^j \exp$$



Conclusion

Motivation: The height function associated to non-intersecting directed polymers defines a random surface that should fall in a new universality class.

Main result: We have shown (with P. Le Doussal) that the stationary measure associated to n non-intersecting directed polymers is an explicit functional of n Brownian motions. It can be shown using a symmetry argument recently employed to study stationary measures of the KPZ equation with boundaries [B.-Le Doussal 2021, B.-Corwin 2022]

For large n, the endpoint measure remains to be studied.

Thank you