Some aspects of the Anderson hamiltonian with white noise in 1D

Laure Dumaz[1], joint works with Cyril Labbé[2]

[1] CNRS, DMA, École Normale Supérieure
[2] LPSM, Université Paris Cité

Coulomb gases and universality
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Ultimate goal : Understand random Schrödinger operators using the link with RMT ?

Schrödinger operator

Differential self-adjoint operator on $\mathbb{R}^d$:

$$u \mapsto -\Delta u + V \cdot u\,.$$

V: potential.

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Interpolation between Laplacian:

$$u \mapsto -\Delta u \,,$$

and multiplication by the potential V:

$$u \mapsto V \cdot u \,.$$

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electron conduction in a crystal.

The potential V models the impurities.

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Main results/conjectures:

- $d = 1$: Anderson localization (result).
- $d = 2$: Anderson localization (conjecture).
- $d = 3$ and above, there is a delocalized phase if V small enough (major conjecture!).

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- **Advantages**: white noise is natural, it is explicit, and it has nice properties.
- **Inconvenient**: it is irregular.

Introduce:

$$\mathcal{H}_L \ : \ u \in \mathcal{D}_L \mapsto -\partial_t^2 u + \xi \cdot u \,.$$

where $\mathcal{D}_L$ subdomain of $L^2(-L/2, L/2)$ with Dirichlet boundary conditions.

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$$\mathcal{H} \ : \ u \in \mathcal{D} \mapsto -\partial_t^2 u + \xi \cdot u \,.$$

where $\mathcal{D}$ subdomain of $L^2(\mathbb{R})$.

Anderson localization

Theorem (L.D., C. Labbé '22+)

When $L \to \infty$, $\mathcal{H}_L$ converges towards $\mathcal{H}$ in the strong resolvent sense.

$\mathcal{H}$ has a pure point spectrum, its eigenfunctions decay exponentially: for every eigenvalue λ,

$$\frac{1}{|t|} \ln \sqrt{\varphi_\lambda(t)^2 + \varphi_\lambda'(t)^2} \to_{|t|\to\infty} -\gamma_\lambda$$

where $\gamma_\lambda > 0$ is the Lyapunov exponent associated to the eigenproblem.

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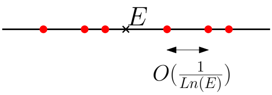
This result was expected but there are some technical issues as white noise is singular.

Study of $\mathcal{H}_L$

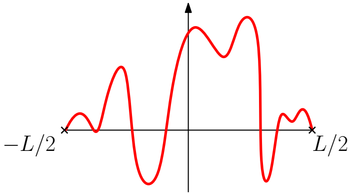
Study the fine statistics of the **spectrum** of $\mathcal{H}_L$ when $L \to \infty$.

Eigenvalues



Random point process on the real line

Eigenvectors



Random real function on $[-L/2, L/2]$

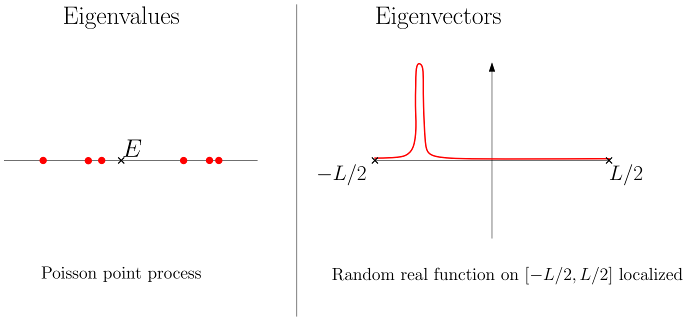
E: energy: it may depend on the size of the system L.

$n(E)$: density of states

(roughly, # of eigenvalues $[E, E+\varepsilon] \simeq n(E) L \varepsilon$)

Goal

Study the **spectrum** of this operator when $L \to \infty$.



Usually for random operators with discrete spectrum, there is a dichotomy:

- Localization of the eigenvectors and Poisson distribution of eigenvalues.

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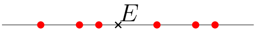
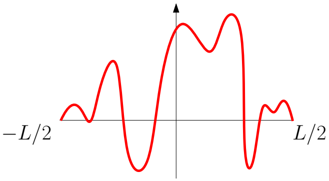


Random point process with repulsion

Eigenvectors



Random real function on $[-L/2, L/2]$ delocalized

Usually for random operators with discrete spectrum, there is a dichotomy:

- Delocalization of the eigenvectors and repulsion of the eigenvalues.

Summary of our results

- **When** $E \ll L$**:**

  Eigenvalues: Poisson, *Eigenvectors:* localized.

- **When** $E = O(L)$ **or** $E \gg L$**:**

  Eigenvalues: repulsion, *Eigenvectors:* delocalized.

Critical Regime $E = O(L)$

Zoom around energies E of order L.

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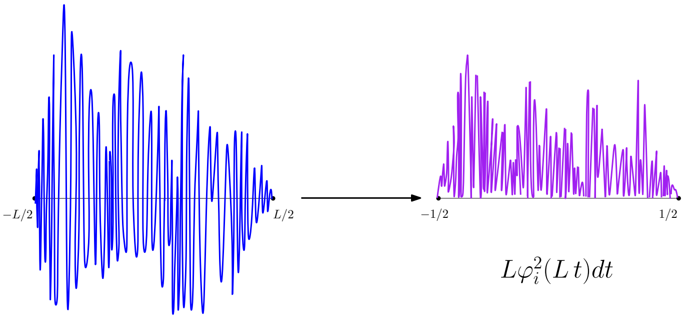
- **Eigenfunctions** :

$$L\varphi_i^2(Lt)dt\,.$$



Eigenfunction φ_i

Delocalization: Critical regime

Fix $\tau > 0$ and let $L/E \to \tau$.

$$\mathcal{Q}_L := \sum_{i \geq 1} \delta_{(Ln(E)(\lambda_i - E), L\varphi_i^2(Lt)dt)},$$

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Theorem (D., Labbé ('21))

$\mathcal{Q}_L$ converges towards a random point process, characterized by coupled diffusions.

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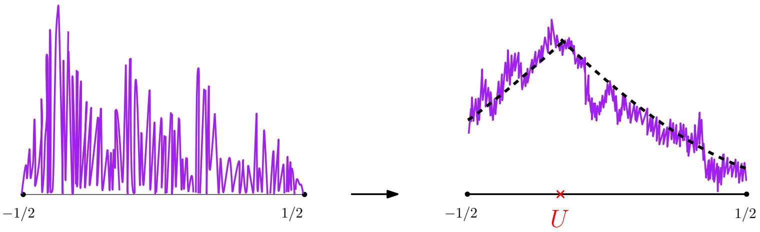
- Typical limiting eigenvectors = exponential of a Brownian motion plus a drift around a uniform point in $[-1/2, 1/2]$ (conjectured for this model by Rifkind and Virág (2018)). **Interaction** between the eigenvectors.

Delocalization: Behavior of eigenvectors



$L\varphi_i^2(L\,t)dt$

$C\ \exp\big(-\frac{\tau}{4}|t-U|+\frac{\sqrt{\tau}}{\sqrt{2}}W(t-U)\big)dt$

Eigenvector associated to an energy near E

$U \sim$ uniform

W two-sided Brownian motion

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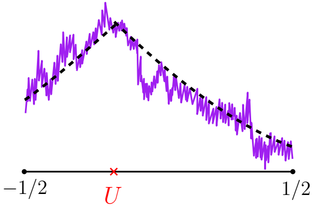

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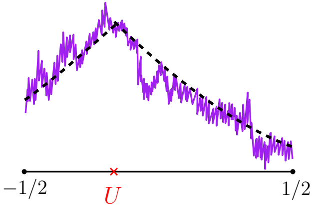

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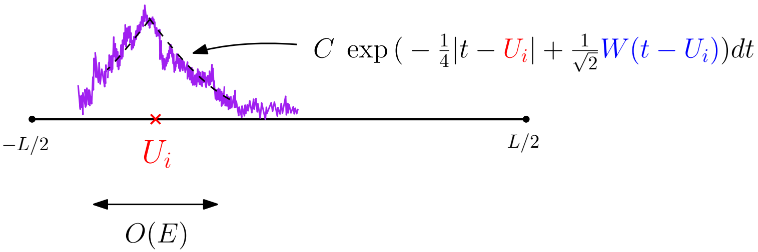
This shape starts to appear in the **localized regime** when $1 \ll E \ll L$ when we zoom around the argmax of the function!

From localization to delocalization

Localized regime for $1 \ll E \ll L$: zoom around the argmax of the function!



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Convergence at the level of **operators**:

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The rescaled (around E) and unitarily changed operator $\mathcal{H}_L$ converges towards the operator

$$\mathtt{CS}_\tau := 2\begin{pmatrix} 0 & -\partial_t \\ \partial_t & 0 \end{pmatrix} + \sqrt{\tau}\begin{pmatrix} d\mathcal{B} + \frac{1}{\sqrt{2}}d\mathcal{W}_1 & \frac{1}{\sqrt{2}}d\mathcal{W}_2 \\ \frac{1}{\sqrt{2}}d\mathcal{W}_2 & d\mathcal{B} - \frac{1}{\sqrt{2}}d\mathcal{W}_1 \end{pmatrix},$$

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- The convergence holds in law in the strong resolvent sense + there is convergence in law of the eigenvalues/eigenvectors.
- Such a form was appearing in the works of Edelman and Sutton (2006) as a conjecture for the bulk of limiting random matrices.

Delocalization: Critical regime

Some remarks:

- The limiting operator acts on $\mathbb{R}^2$ valued functions: The initial space is **enlarged**. $\rightarrow$ transformation of $\mathcal{H}_L$ is non trivial.

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- $\mathtt{CS}_\tau$ is not properly defined in this form. An elegant way to define it is though *Dirac operators*, which appeared recently in Valkó and Virág (2016) for the limit of many random matrix models.

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- $\mathtt{CS}_\tau$ is not properly defined in this form. An elegant way to define it is though *Dirac operators*, which appeared recently in Valkó and Virág (2016) for the limit of many random matrix models.
- Our approach could probably be extended to other models (bulk of tridiagonal matrices or other differential operators).

Link with the operators of Valkó and Virág

Dirac operators: differential self-adjoint operator of the form

$$\tau_R \ : \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto 2\, R(t)^{-1} J \partial_t \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$R(t)$: 2×2 symmetric positive matrix of $\det = 1$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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Valkó and Virág ('16) proved that the spectrum of those operators for a well-chosen random R corresponds to the spectrum of various famous RMT (Sine_β, Bessel_β etc.).

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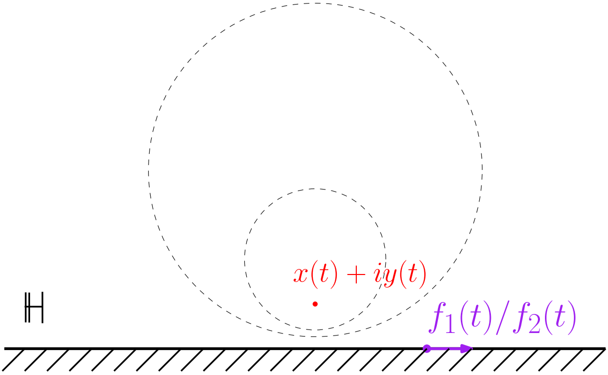

Link with the operators of Valkó and Virág

Theorem (D.,Labbé, 2021)

The operator $\mathtt{CS}_\tau$ is unitarily equivalent to a Dirac operator associated to a hyperbolic Brownian motion of variance $\tau/2$.

Unitary map is random (non trivial).
Boundary conditions are complicated but explicit.

Some ideas for the proofs in the localized regime

Eigenvalue equation

Eigenvalue equation for $\mathcal{H}_L$ defined on $[-L/2, L/2]$:

$$-\varphi'' + \xi \cdot \varphi = \lambda\varphi$$

with $\varphi(-L/2) = 0$ (without any condition on $\varphi(L/2)$).

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To keep in mind: it is much easier to analyse the solution of the ODE than the solution of the eigenvalue problem: λ is random and depends on the whole potential ξ.

Eigenvalue equation

One can also impose first $\hat{\varphi}(L/2) = 0$ and solve

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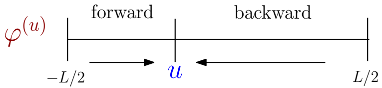
Concatenation forward/backward

Key idea: Use *forward solution* φ_λ on the time-interval $[-L/2, u]$ and then *backward solution* $\hat{\varphi}_\lambda$ on $[u, L/2]$ for some well-chosen u.
$\rightarrow$ Concatenation is $\varphi^{(u)}$.



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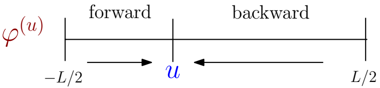

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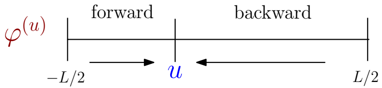

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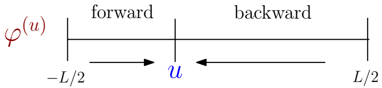

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It helps A LOT because it is **much easier** to analyze the forward or backward solution of the ODE than the eigenvalue equation (when λ eigenvalue, λ is random and depends on the whole potential ξ!).

Localization: A key formula

Proposition (Goldsheid Molchanov Pastur formula)

For all continuous, compactly supported on the first variable, and bounded G:

$$\mathbb{E}\Big[\sum_{\lambda \text{ eigenvalue}} G(\lambda, \varphi_\lambda)\Big]$$

$$= \int_{\lambda\in\mathbb{R}} \int_{-L/2}^{L/2} \int_0^{\pi} \sin^2(\theta) p_\lambda(\theta) p_\lambda(\pi-\theta) \mathbb{E}\Big[G\Big(\lambda, \frac{\varphi_\lambda^{(u)}}{||\varphi_\lambda^{(u)}||_2}\Big)\Big] d\lambda du d\theta,$$

where

- $\varphi^{(u)}$ *is the concatenation of the forward process and backward process at time u.*
- $p_\lambda(\theta)$ *transition probability of θ_λ "phase function" (argument of $\varphi'_\lambda + i\varphi_\lambda$).*

THANK YOU!

Strategy to prove convergence towards a Poisson point process when you know localization

Let $\Delta = [E - h/(Ln(E)), E + h/(Ln(E))]$ (E fixed) and denote $N(\Delta) = \#$ eigenvalues in Δ.

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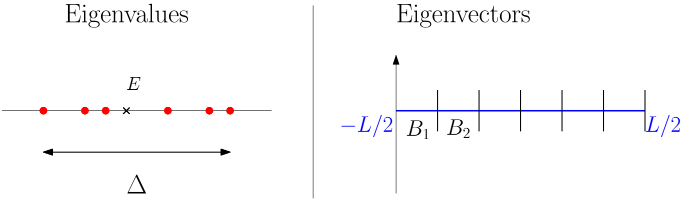
Divide $[-L/2, L/2]$ into small boxes B_i, $i = 1, \cdots, N$ of same length.



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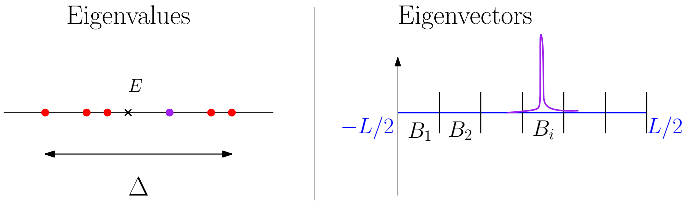

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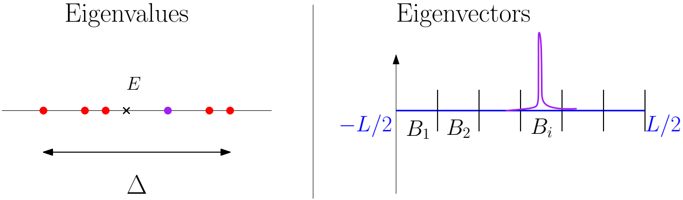

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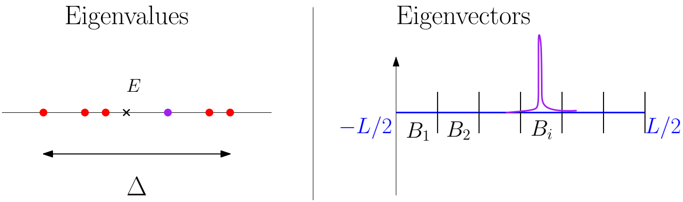

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(C) $\sum_i \mathbb{P}[N_i(\Delta) \geq 2] \to 0$.