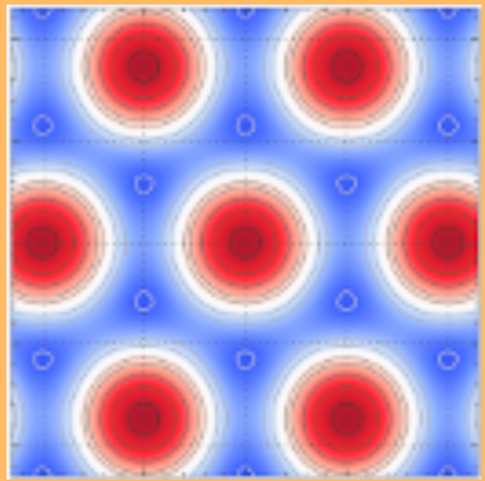
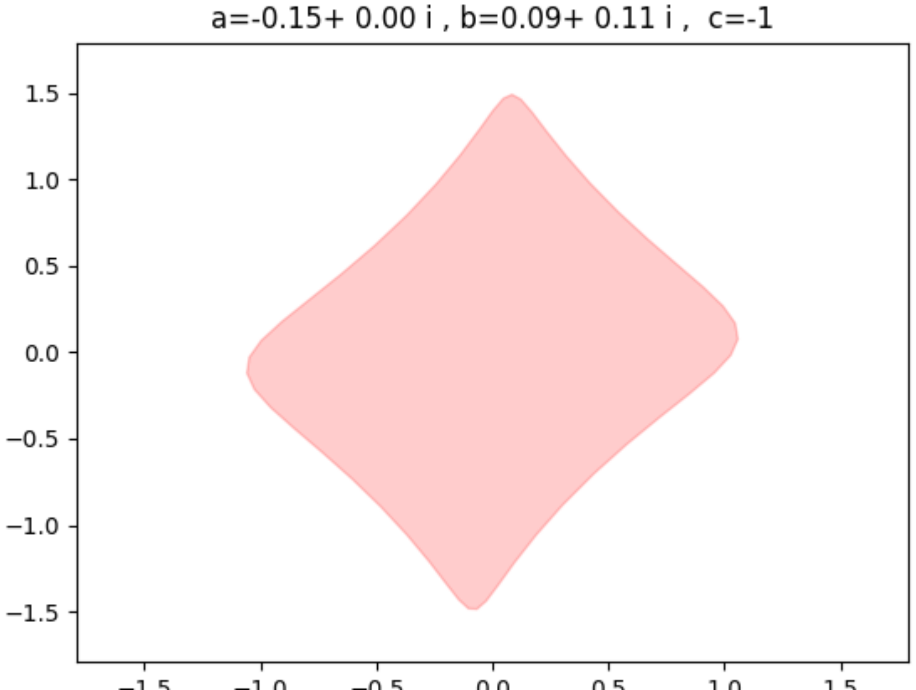


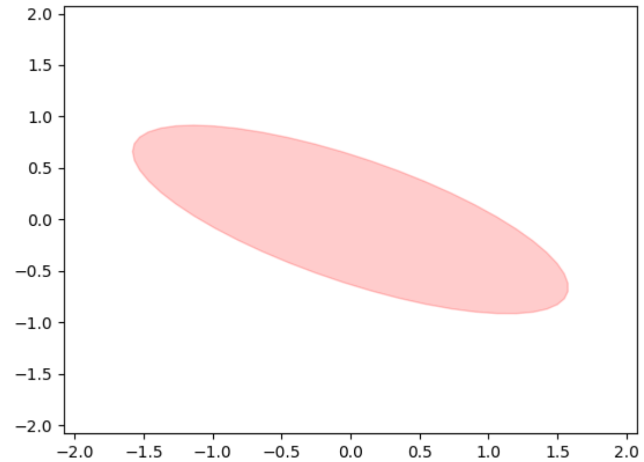
Coulomb Gases

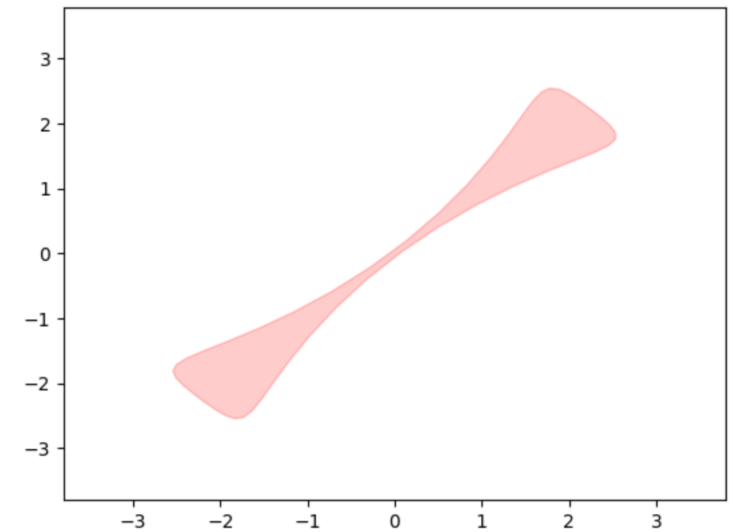
To algebraic geometry

Paris, dec 2022

B. Eynard *IPHT CEA Saclay, CRM*







Plan

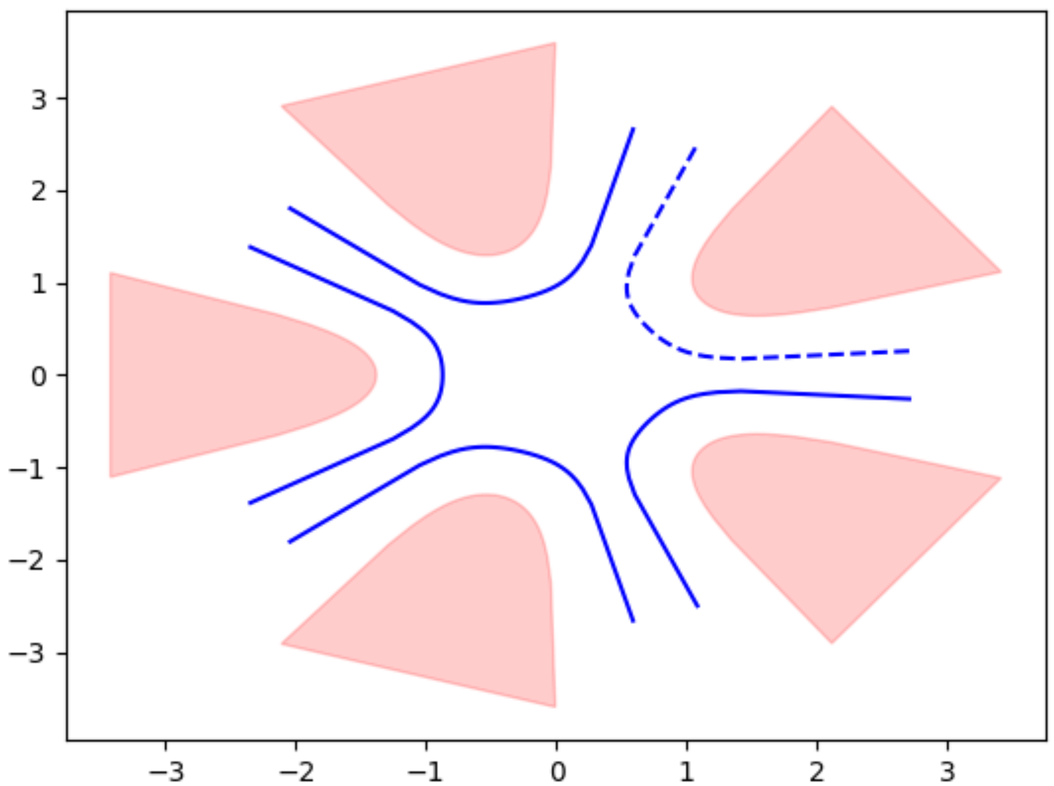
- **Random Matrices**
  - **Limit density algebraic**

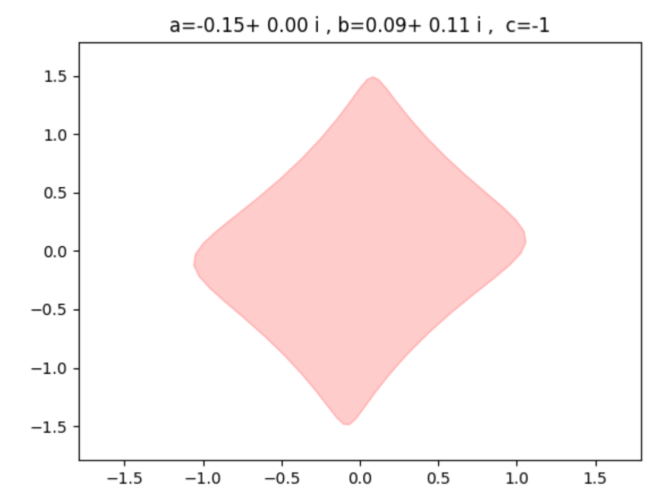
- **Coulomb gaz**
  - **Eigenvalues -> Coulomb gaz**

- **Limit shapes**

- **Asymptotic expansion**

- **Conclusion**





Plan

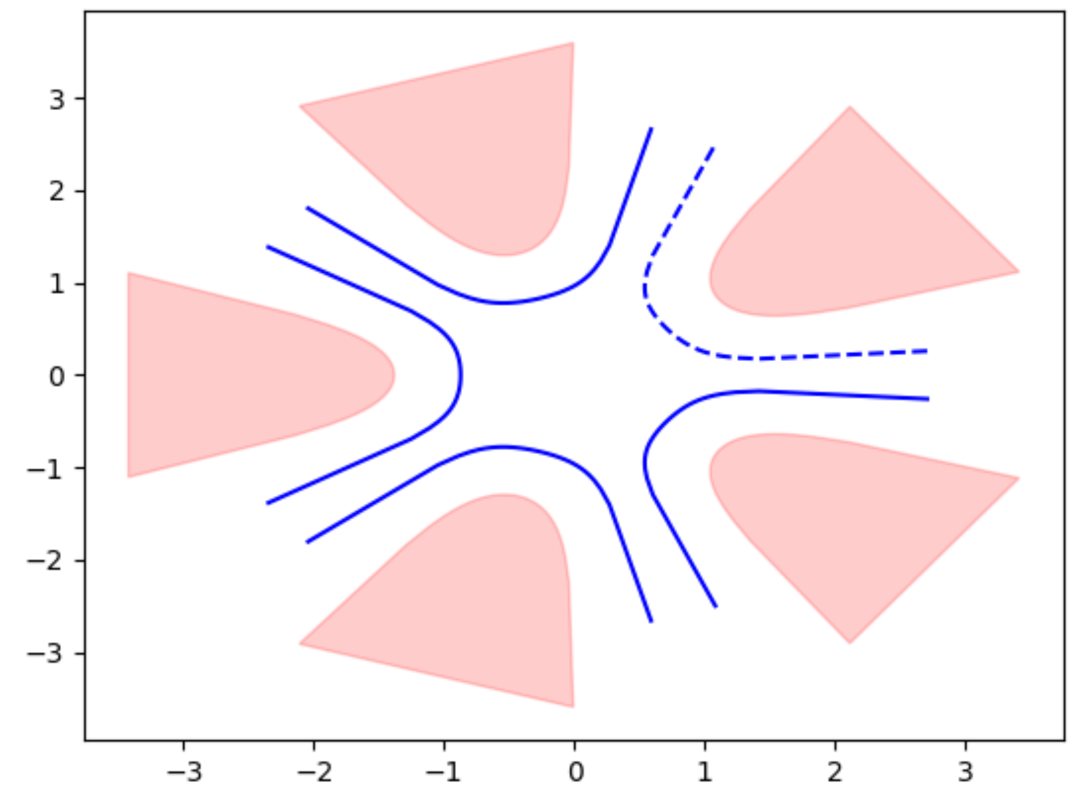
- **Random Matrices**
  - **Limit density algebraic**

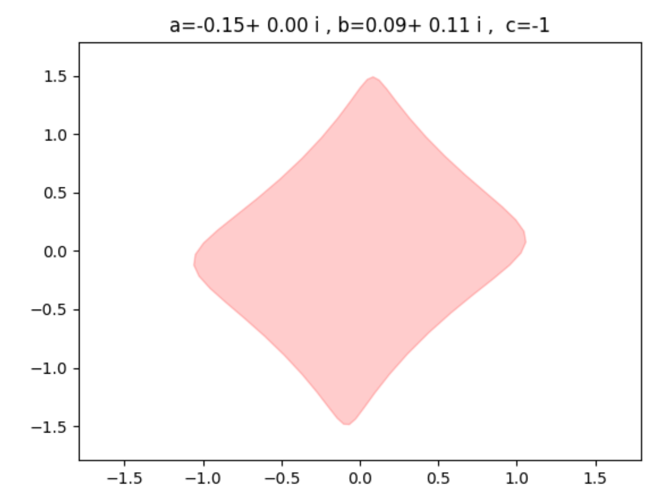
- **Coulomb gaz**
  - **Eigenvalues -> Coulomb gaz**

- **Limit shapes**

- **Asymptotic expansion**

- **Conclusion**





Random matrices

Random matrices

- Hermitian matrix $M \in H_N$

Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

  Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

  Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

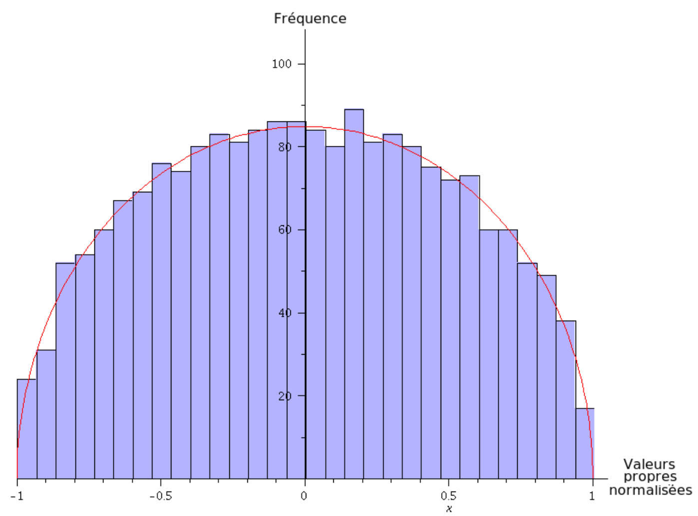
  Example Gaussian : $V(x) = \frac{x^2}{2}$

Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Example Gaussian : $V(x) = \frac{x^2}{2}$

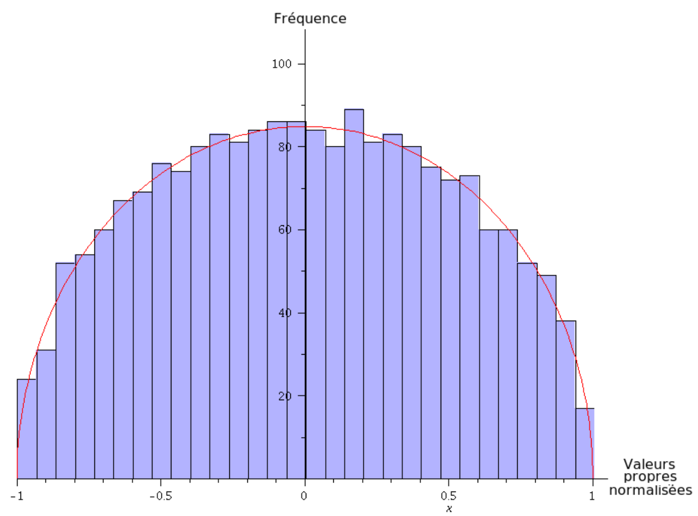


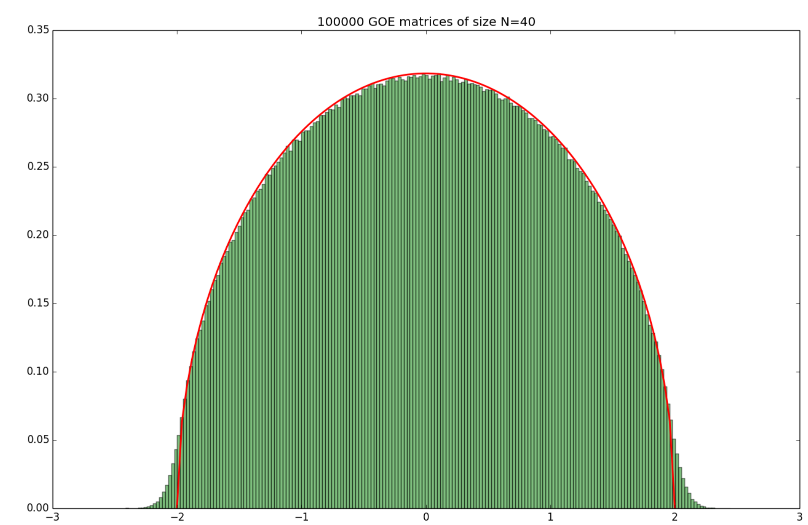
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

  Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

  Example Gaussian : $V(x) = \frac{x^2}{2}$



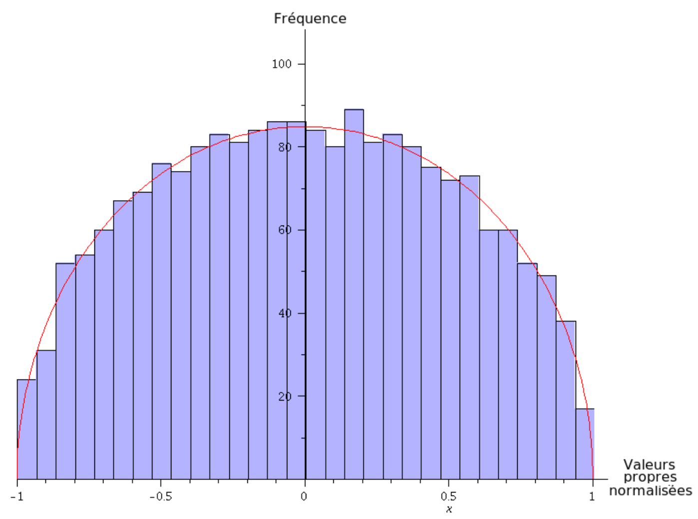


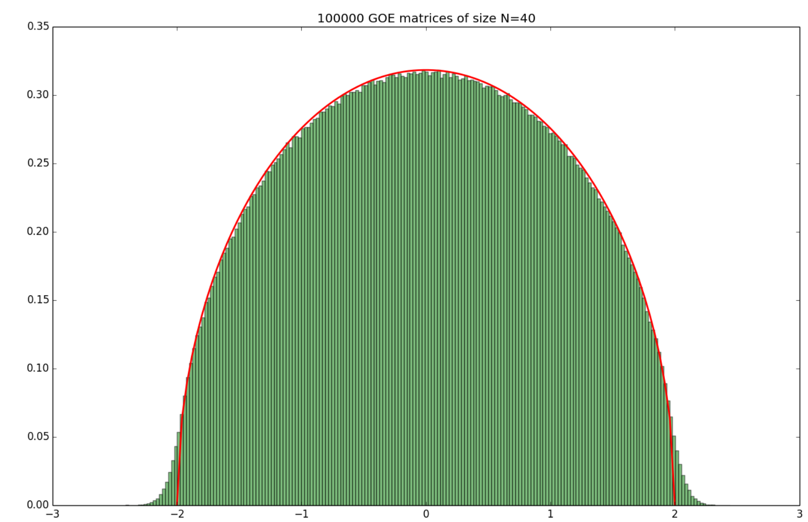
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Example Gaussian : $V(x) = \frac{x^2}{2}$





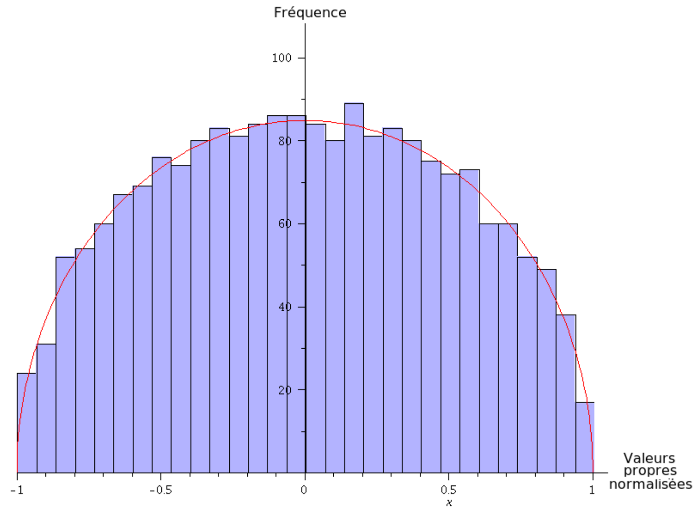
- Large N Limit of spectral density : $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\ \mathbf{1}_{[-2,2]}$

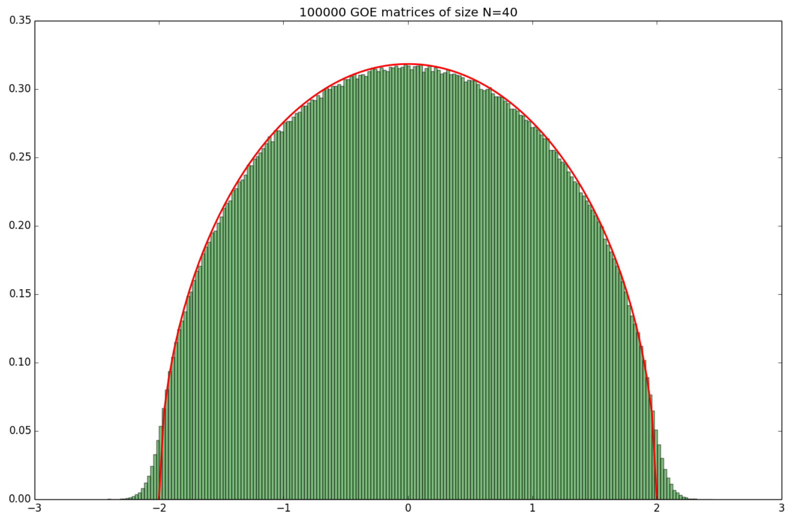
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $\quad V(x) \in \mathbb{C}[x]$

Example Gaussian : $V(x) = \frac{x^2}{2}$





- Large N Limit of spectral density : $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\ \mathbf{1}_{[-2,2]}$

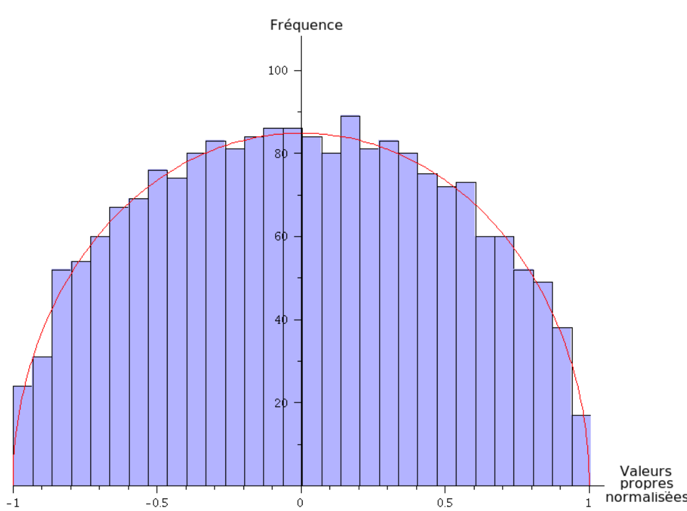
Algebraic !

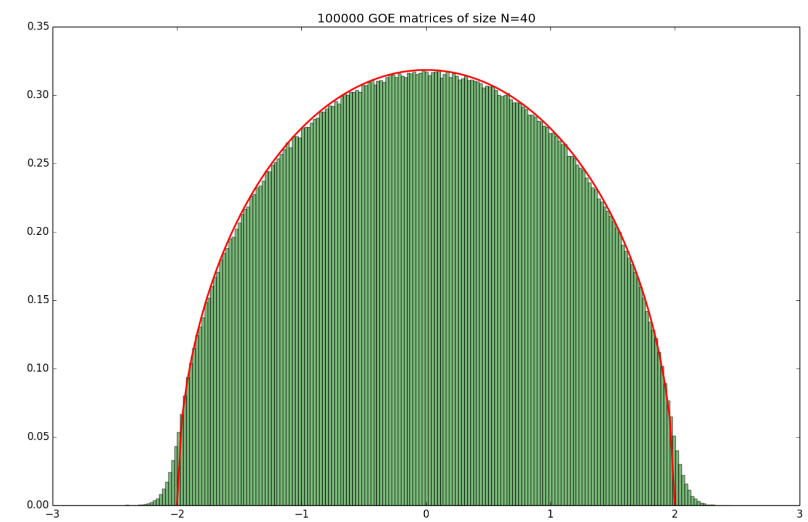
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

  Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

  Example Gaussian : $V(x) = \frac{x^2}{2}$





- Large N Limit of spectral density : $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\ \mathbf{1}_{[-2,2]}$ $\rho(x) = \frac{1}{2\pi i} y$

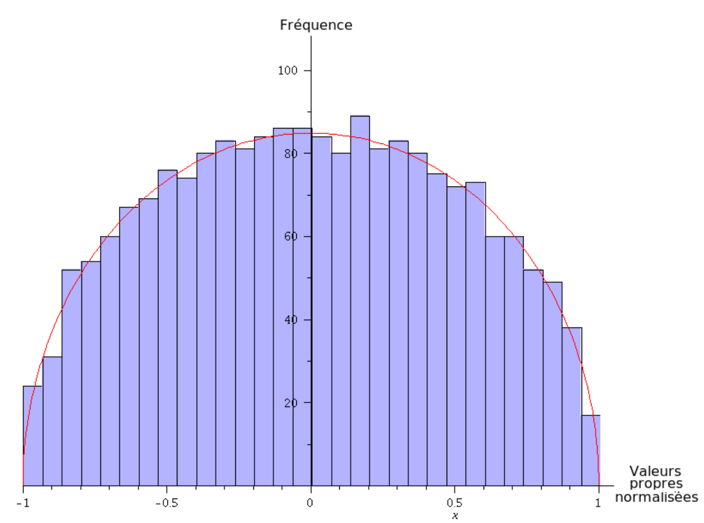
Algebraic !

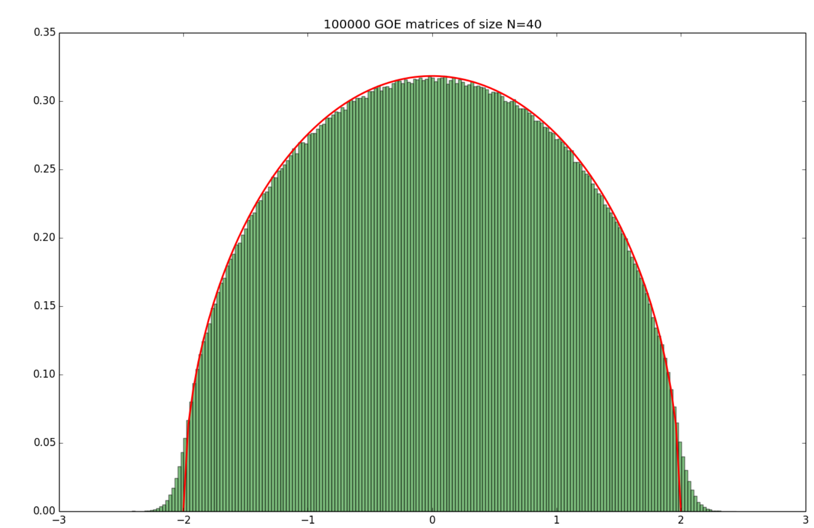
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Example Gaussian : $V(x) = \frac{x^2}{2}$





- Large N Limit of spectral density : $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \; \mathbf{1}_{[-2,2]}$ $\rho(x) = \frac{1}{2\pi i} y$

$y^2 - x^2 + 4 = 0$

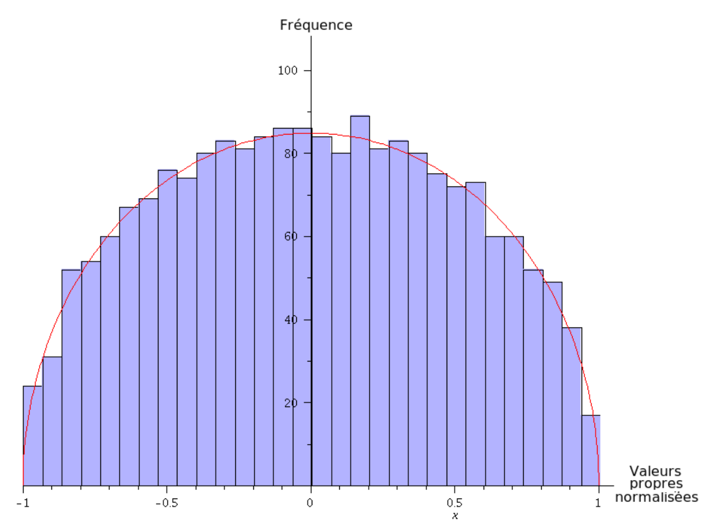
Algebraic !

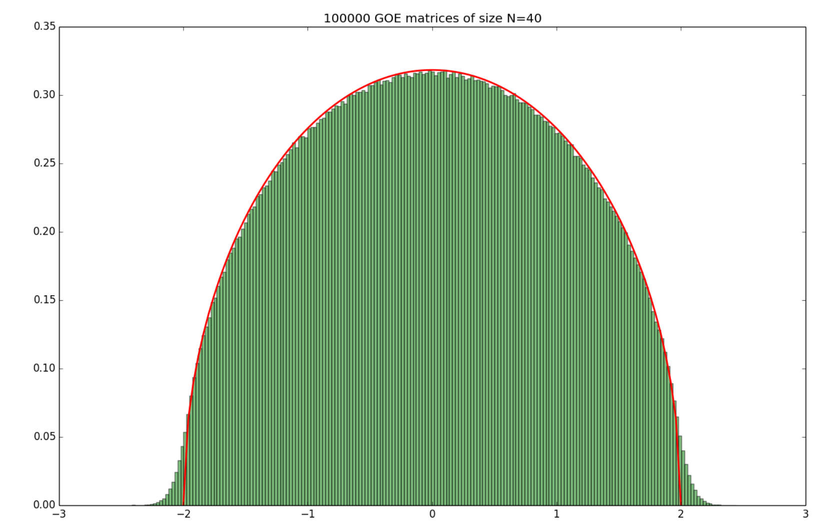
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Example Gaussian : $V(x) = \frac{x^2}{2}$





- Large N Limit of spectral density : $\rho(x) = \frac{1}{2\pi}\sqrt{4-x^2}\ \mathbf{1}_{[-2,2]}$ $\rho(x) = \frac{1}{2\pi i} y$

$y^2 - x^2 + 4 = 0$

Algebraic !

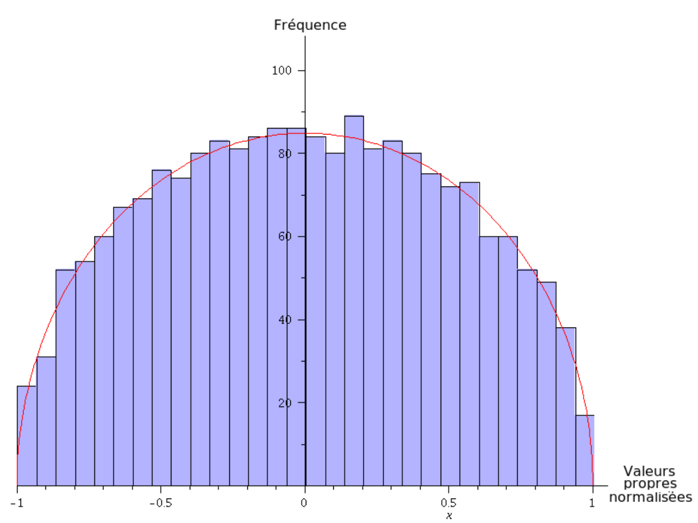
$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\operatorname{Tr}(x-M)^{-1}\right) = \frac{1}{2}\left(x - \sqrt{x^2-4}\right) = \frac{1}{2}(x-y)$$

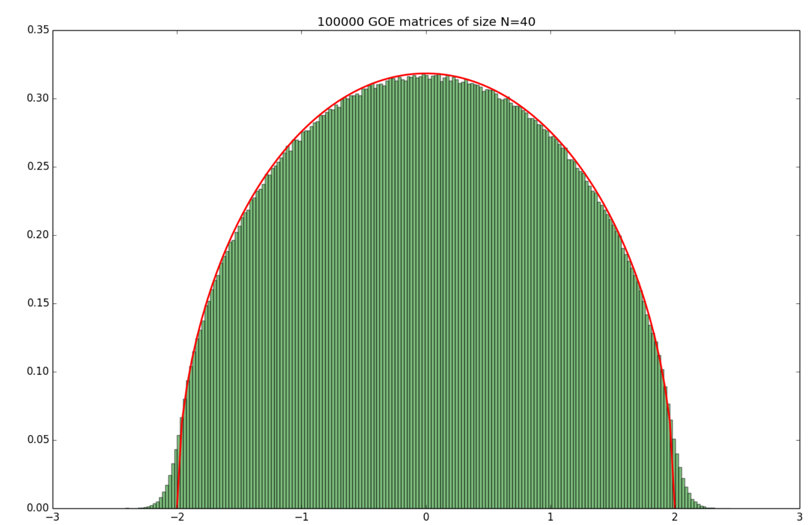
Random matrices

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

  Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \,\mathrm{Tr}\, V(M)} dM$ $V(x) \in \mathbb{C}[x]$

  Example Gaussian : $V(x) = \frac{x^2}{2}$





- Large N Limit of spectral density : $\rho(x) = \frac{1}{2\pi} \sqrt{4 - x^2}\ \mathbf{1}_{[-2,2]}$ $\rho(x) = \frac{1}{2\pi i} y$

$y^2 - x^2 + 4 = 0$

Algebraic !

$$\lim_{N\to\infty} \mathbb{E}\left(\frac{1}{N}\mathrm{Tr}(x-M)^{-1}\right) = \frac{1}{2}\left(x - \sqrt{x^2-4}\right) = \frac{1}{2}(x-y)$$

Eigenvalues

Eigenvalues

Random matrices and Coulomb Gases

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Induced measure for eigenvalues : $M = U\Lambda U^\dagger$ $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ $U \in U(N)/U(1)^N$

Real eigenvalues $\lambda_i \in \mathbb{R}$ $\Lambda \in \mathbb{R}^N$

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Induced measure for eigenvalues : $M = U \Lambda U^\dagger$ $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ $U \in U(N)/U(1)^N$

Real eigenvalues $\lambda_i \in \mathbb{R}$ $\Lambda \in \mathbb{R}^N$

$$d\mu(M) \to d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} d\lambda_i$$

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \,\mathrm{Tr}\, V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Induced measure for eigenvalues : $M = U\Lambda U^\dagger$ $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $U \in U(N)/U(1)^N$

Real eigenvalues $\lambda_i \in \mathbb{R}$ $\Lambda \in \mathbb{R}^N$

$$d\mu(M) \to d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^N e^{-NV(\lambda_i)} d\lambda_i$$

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^N V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \,\mathrm{Tr}\, V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Induced measure for eigenvalues : $M = U \Lambda U^\dagger$ $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $U \in U(N)/U(1)^N$

Real eigenvalues $\lambda_i \in \mathbb{R}$ $\Lambda \in \mathbb{R}^N$

$$d\mu(M) \to d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^N e^{-NV(\lambda_i)} d\lambda_i$$

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^N V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

Coulomb repulsion
charge 1

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Induced measure for eigenvalues : $M = U \Lambda U^\dagger$ $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ $U \in U(N)/U(1)^N$

Real eigenvalues $\lambda_i \in \mathbb{R}$ $\Lambda \in \mathbb{R}^N$

$$d\mu(M) \to d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^N e^{-NV(\lambda_i)} d\lambda_i$$

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^N V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

Potential

Coulomb repulsion
charge 1

Random matrices and Coulomb Gases

- Hermitian matrix $M \in H_N$ Lebesgue measure $dM = \prod_i dM_{i,i} \prod_{i<j} d\Re M_{i,j} \wedge d\Im M_{i,j}$

Probability law of Boltzman type : $d\mu(M) = \frac{1}{Z} e^{-N \operatorname{Tr} V(M)} dM$ $V(x) \in \mathbb{C}[x]$

Induced measure for eigenvalues : $M = U\Lambda U^\dagger$ $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ $U \in U(N)/U(1)^N$

Real eigenvalues $\lambda_i \in \mathbb{R}$ $\Lambda \in \mathbb{R}^N$

$$d\mu(M) \to d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^N e^{-NV(\lambda_i)} d\lambda_i$$

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^N V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

Potential

Coulomb repulsion charge 1

Normal Random matrices and Coulomb Gases

Normal Random matrices and Coulomb Gases

- **Normal matrix :**

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $M = U \Lambda U^{\dagger}$ $U \in U(N)/U(1)^N$

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $M = U \Lambda U^\dagger$ $U \in U(N)/U(1)^N$

  $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\lambda_i \in \gamma$ $\Lambda \in \gamma^N$

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $M = U\Lambda U^\dagger$ $U \in U(N)/U(1)^N$

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\lambda_i \in \gamma$ $\Lambda \in \gamma^N$

Where γ can be: any measurable space: Polish subsepace of $\mathbb{R}$
Jordan arc in $\mathbb{C}$ (Suslin space)

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $M = U\Lambda U^\dagger$ $U \in U(N)/U(1)^N$

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\lambda_i \in \gamma$ $\Lambda \in \gamma^N$

Where γ can be: any measurable space: Polish subsepace of $\mathbb{R}$
Jordan arc in $\mathbb{C}$ (Suslin space)

Measure (complex) of Boltzman type for eigenvalues :

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $M = U\Lambda U^\dagger$ $U \in U(N)/U(1)^N$

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\lambda_i \in \gamma$ $\Lambda \in \gamma^N$

Where γ can be: any measurable space: Polish subsepace of $\mathbb{R}$
Jordan arc in $\mathbb{C}$ (Suslin space)

Measure (complex) of Boltzman type for eigenvalues :

$$d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} d\lambda_i \qquad \Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $M = U\Lambda U^\dagger$ $U \in U(N)/U(1)^N$

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\lambda_i \in \gamma$ $\Lambda \in \gamma^N$

Where γ can be: any measurable space: Polish subsepace of $\mathbb{R}$
Jordan arc in $\mathbb{C}$ (Suslin space)

Measure (complex) of Boltzman type for eigenvalues :

$$d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} d\lambda_i \qquad \Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight :

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

$$d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$$

Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $\quad M = U\Lambda U^\dagger$ $\quad U \in U(N)/U(1)^N$

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\quad \lambda_i \in \gamma$ $\quad \Lambda \in \gamma^N$

Where γ can be: any measurable space: Polish subsepace of $\mathbb{R}$
Jordan arc in $\mathbb{C}$ (Suslin space)

Measure (complex) of Boltzman type for eigenvalues :

$$d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} d\lambda_i \qquad \Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight :

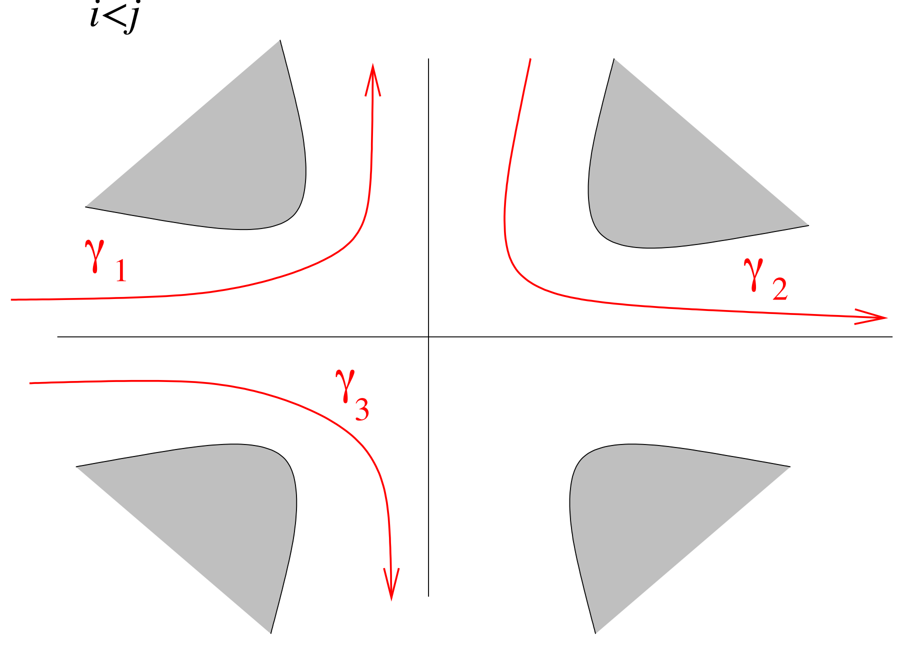
$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

$$d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$$

Linear combinations of paths:

$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ $\quad \lambda_i \in \gamma$ $\quad \Lambda \in \gamma^N$

$$\gamma = \sum_{i=1}^{\deg V'} c_i \gamma_i$$



Normal Random matrices and Coulomb Gases

- **Normal matrix :** $M \in H_N(\gamma)$ $\quad M = U \Lambda U^\dagger$ $\quad U \in U(N)/U(1)^N$

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \qquad \lambda_i \in \gamma \qquad \Lambda \in \gamma^N$$

Where γ can be: any measurable space: Polish subsepace of $\mathbb{R}$
Jordan arc in $\mathbb{C}$ (Suslin space)

Measure (complex) of Boltzman type for eigenvalues :

$$d\mu(\Lambda) = \Delta(\Lambda)^2 \prod_{i=1}^{N} e^{-NV(\lambda_i)} d\lambda_i \qquad \Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j) = \det_{i,j} \lambda_j^{N-i}$$

Rewritten as a Boltzman weight :

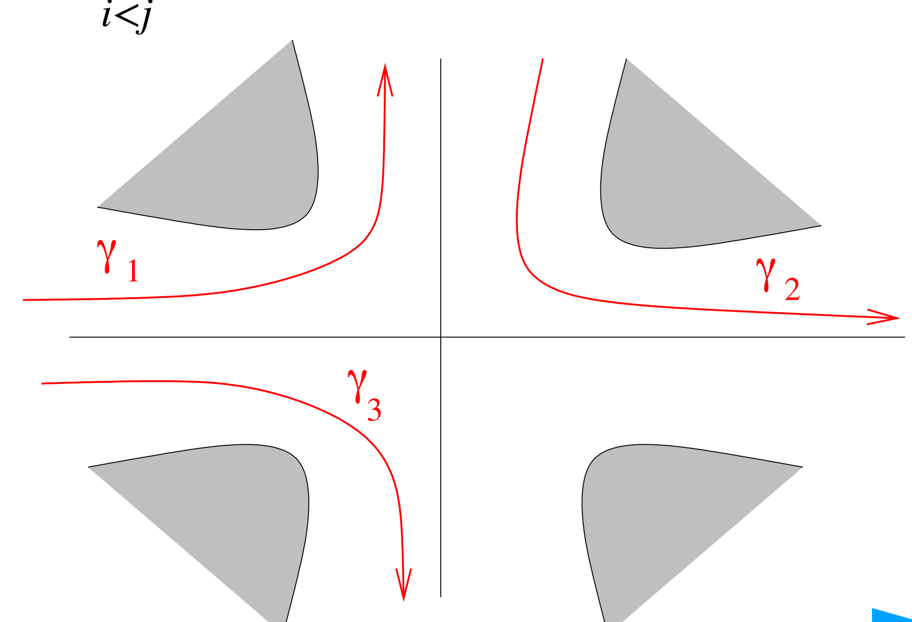
$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$$

$$d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$$

Linear combinations of paths:

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \qquad \lambda_i \in \gamma \qquad \Lambda \in \gamma^N$$

$$\gamma = \sum_{i=1}^{\deg V'} c_i \gamma_i$$



Hermitian 2-matrix model

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N \,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N \operatorname{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues :

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger \quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \quad U_i \in U(N)/U(1)^N$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :**

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

$M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad \Lambda_1 \in \gamma^N$ $\quad \Lambda_2 \in \tilde{\gamma}^N$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

  $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad \Lambda_1 \in \gamma^N$ $\quad \Lambda_2 \in \tilde{\gamma}^N$

Measures for eigenvalues :

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

$M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad \Lambda_1 \in \gamma^N$ $\quad \Lambda_2 \in \tilde{\gamma}^N$

Measures for eigenvalues :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger \quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

$$M_i = U_i \Lambda_i U_i^\dagger \qquad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N$$

Measures for eigenvalues :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Partition function:

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger \quad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2)\det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

$$M_i = U_i \Lambda_i U_i^\dagger \qquad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N$$

Measures for eigenvalues :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N\,\mathrm{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger \qquad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2)\det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

$$M_i = U_i \Lambda_i U_i^\dagger \qquad \Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \qquad \Lambda_2 \in \tilde{\gamma}^N$$

Measures for eigenvalues :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i} \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$

Hermitian 2-matrix model

- **2 Hermitian matrices** $M_1, M_2 \in H_N^2$

  Probability law of Boltzman type : $$d\mu(M_1, M_2) = \frac{1}{Z} e^{-N \operatorname{Tr}(V_1(M_1)+V_2(M_2)-cM_1M_2)} dM$$

Induced measure for eigenvalues : $M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \operatorname{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad U_i \in U(N)/U(1)^N$

$$d\mu(M_1, M_2) \to d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det\left(e^{cN\lambda_{1,i}\lambda_{2,j}}\right) \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

- **2 Normal matrices :** $M_1, M_2 \in H_N(\gamma) \times H_N(\tilde{\gamma})$

$M_i = U_i \Lambda_i U_i^\dagger$ $\quad \Lambda_i = \operatorname{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N})$ $\quad \Lambda_1 \in \gamma^N$ $\quad \Lambda_2 \in \tilde{\gamma}^N$

Measures for eigenvalues :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i} \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$

Normal 2-matrix model

Normal 2-matrix model

2 diagonal matrices :

Normal 2-matrix model

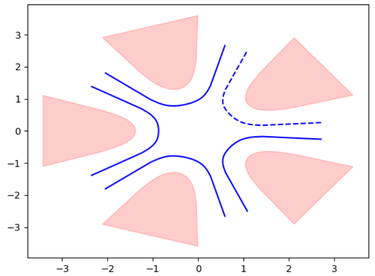
2 diagonal matrices :

$$\Lambda_i = \text{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$

Normal 2-matrix model

2 diagonal matrices :

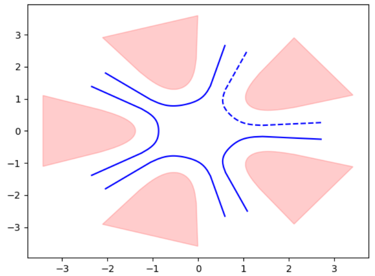
$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$

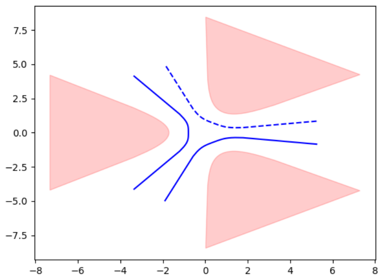


Normal 2-matrix model

2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1},\ldots,\lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1,\Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$

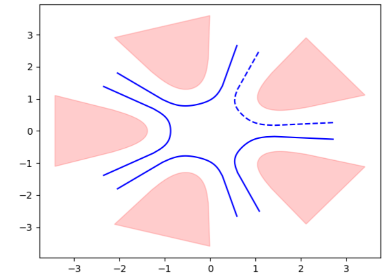


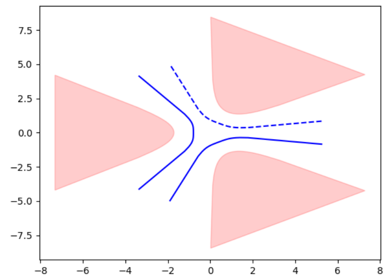


Normal 2-matrix model

2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



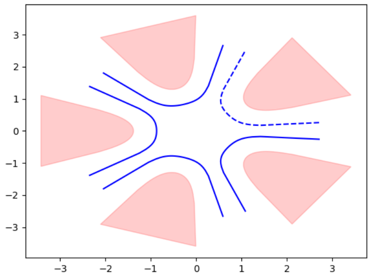


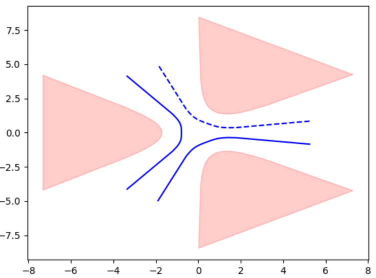
Measure :

Normal 2-matrix model

2 diagonal matrices :

$$\Lambda_i = \text{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$





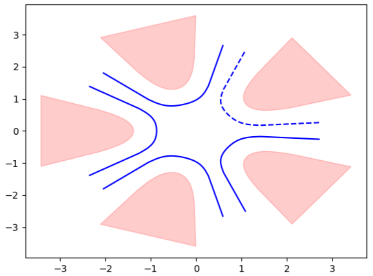
Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Normal 2-matrix model

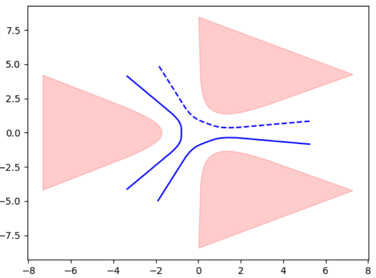
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

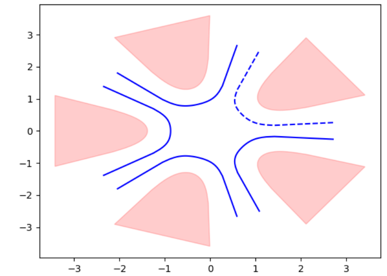


Partition function:

Normal 2-matrix model

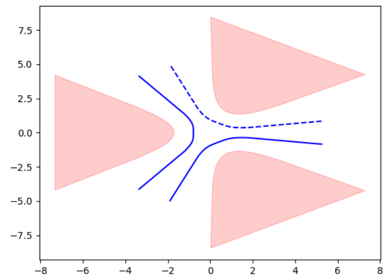
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \qquad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



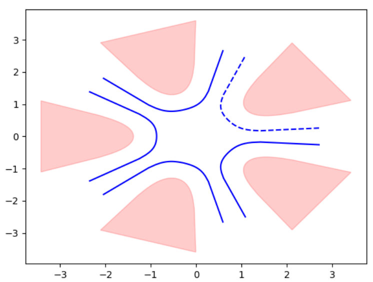
Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Normal 2-matrix model

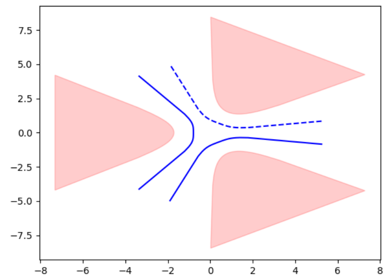
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

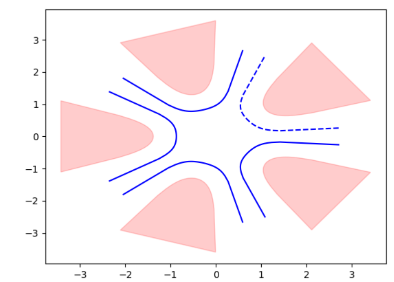
$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Generalization:

Normal 2-matrix model

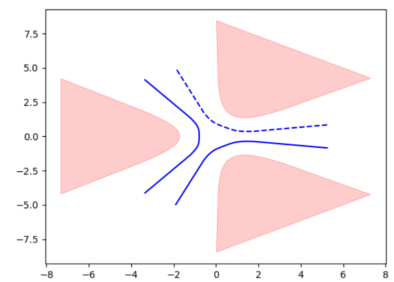
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

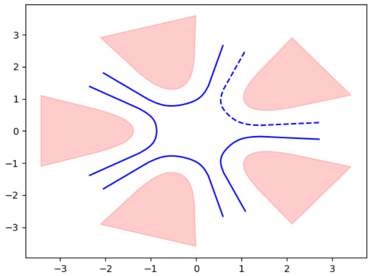
$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Generalization: $(\Lambda_1, \Lambda_2) \in \Gamma$

Normal 2-matrix model

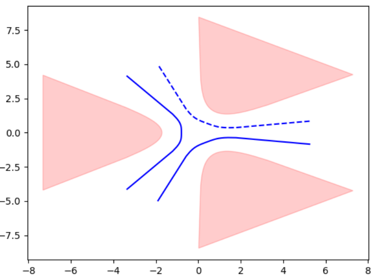
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \qquad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

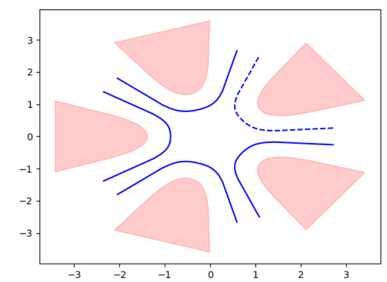
$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2)\det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2}\prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Generalization: $(\Lambda_1, \Lambda_2) \in \Gamma$ basis : linear combination $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$

Normal 2-matrix model

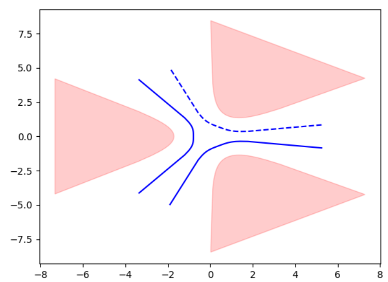
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \qquad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

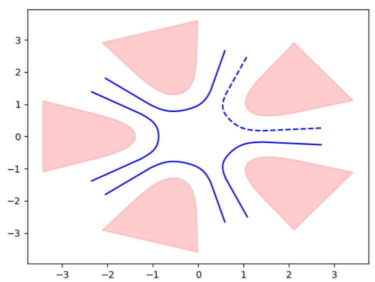
Generalization: $(\Lambda_1, \Lambda_2) \in \Gamma$ basis : linear combination $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$

Generalization:

Normal 2-matrix model

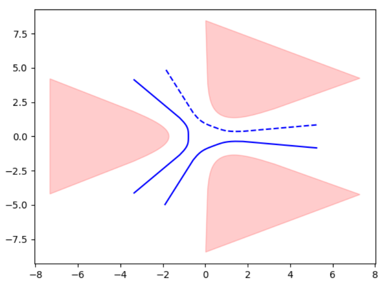
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \qquad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

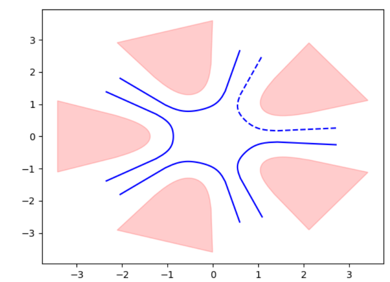
Generalization: $(\Lambda_1, \Lambda_2) \in \Gamma$ basis : linear combination $\Gamma = (\sum_{i,j} c_{i,j} \gamma_i \times \bar{\gamma}_j)^N$

Generalization: $\Gamma =$ any homology class Lagrangian of dimension $2N$ of $\mathbb{C}^{2N}$

Normal 2-matrix model

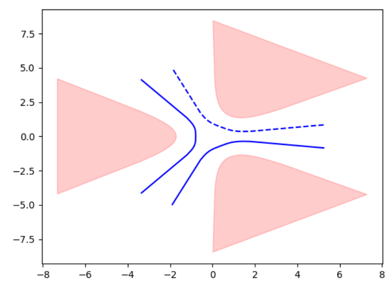
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \quad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Generalization: $(\Lambda_1, \Lambda_2) \in \Gamma$ basis : linear combination $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$

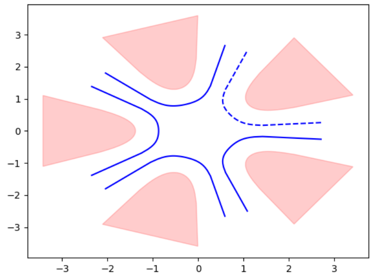
Generalization: $\Gamma =$ any homology class Lagrangian of dimension $2N$ of $\mathbb{C}^{2N}$

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Normal 2-matrix model

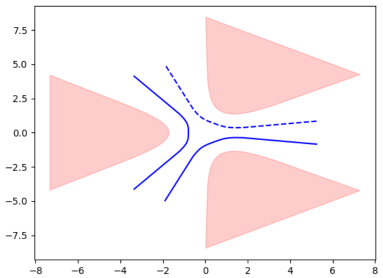
2 diagonal matrices :

$$\Lambda_i = \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,N}) \qquad \Lambda_1 \in \gamma^N \qquad \Lambda_2 \in \tilde{\gamma}^N \qquad (\Lambda_1, \Lambda_2) \in \Gamma \qquad \Gamma = \gamma^N \times \tilde{\gamma}^N$$



Measure :

$$d\mu(\Lambda_1, \Lambda_2) = \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$



Partition function:

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Generalization: $(\Lambda_1, \Lambda_2) \in \Gamma$ basis : linear combination $\Gamma = (\sum_{i,j} c_{i,j} \gamma_i \times \bar{\gamma}_j)^N$

Generalization: $\Gamma =$ any homology class Lagrangian of dimension $2N$ of $\mathbb{C}^{2N}$

$$Z = \int_\Gamma d\mu(\Lambda_1, \Lambda_2) = \int_\Gamma \Delta(\Lambda_1)\Delta(\Lambda_2) \det e^{cN\lambda_{1,i}\lambda_{2,j}} \prod_{j=1}^{2} \prod_{i=1}^{N} e^{-NV_j(\lambda_{j,i})} d\lambda_{j,i}$$

Complex matrix model

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda \ , \ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\ ,\ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda \ , \ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $d\mu(\Lambda_1, \Lambda_2)$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\ ,\ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $$d\mu(\Lambda_1, \Lambda_2) = |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))}\ d^2\lambda_i$$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda \; , \; \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $$d\mu(\Lambda_1, \Lambda_2) \; = |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \; \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))} \; d^2\lambda_i$$

$$Z = N! \int_\Gamma \prod_{i<j} |\lambda_i - \lambda_j|^2 \; \prod_{i=1}^{N} e^{-N(2\Re(V_1(\lambda_i))-c|\lambda_i|^2)} \; d^2\lambda_i$$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x,y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\,,\ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $$d\mu(\Lambda_1, \Lambda_2) = |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))}\ d^2\lambda_i$$

$$Z = N! \int_\Gamma \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_{i=1}^{N} e^{-N(2\mathfrak{R}(V_1(\lambda_i)) - c|\lambda_i|^2)}\ d^2\lambda_i$$

- **Normal matrices** with homology class: $\Gamma = \{\gamma \times \bar{\gamma}\}^N$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\ ,\ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $d\mu(\Lambda_1, \Lambda_2) \ = |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))} \ d^2\lambda_i$

$$Z = N! \int_\Gamma \prod_{i<j} |\lambda_i - \lambda_j|^2 \prod_{i=1}^{N} e^{-N(2\Re(V_1(\lambda_i))-c|\lambda_i|^2)} \ d^2\lambda_i$$

- **Normal matrices** with homology class: $\Gamma = \{\gamma \times \bar{\gamma}\}^N$

  Or more generally with linear combinations $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \tilde{\gamma}_j)^N$

  Case $\tilde{\gamma}_i = \bar{\gamma}_i$ $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$ $\bar{c}_{i,j} = c_{j,i}$

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\,,\;\; \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $d\mu(\Lambda_1,\Lambda_2) \;= |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))}\; d^2\lambda_i$

$$Z = N! \int_\Gamma \prod_{i<j} |\lambda_i - \lambda_j|^2 \;\prod_{i=1}^{N} e^{-N(2\Re(V_1(\lambda_i)) - c|\lambda_i|^2)}\; d^2\lambda_i$$

- **Normal matrices** with homology class: $\Gamma = \{\gamma \times \bar{\gamma}\}^N$

  Or more generally with linear combinations $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \tilde{\gamma}_j)^N$

  Case $\tilde{\gamma}_i = \bar{\gamma}_i$ $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$ $\bar{c}_{i,j} = c_{j,i}$

- This is the 2-dimensional complex Coulomb gaz

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\ ,\ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $d\mu(\Lambda_1, \Lambda_2) = |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))}\ d^2\lambda_i$

$$Z = N! \int_\Gamma \prod_{i<j} |\lambda_i - \lambda_j|^2 \ \prod_{i=1}^{N} e^{-N(2\mathfrak{R}(V_1(\lambda_i)) - c|\lambda_i|^2)}\ d^2\lambda_i$$

- **Normal matrices** with homology class: $\Gamma = \{\gamma \times \bar{\gamma}\}^N$

  Or more generally with linear combinations $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \tilde{\gamma}_j)^N$

  Case $\tilde{\gamma}_i = \bar{\gamma}_i$ $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$ $\bar{c}_{i,j} = c_{j,i}$

- This is the 2-dimensional complex Coulomb gaz

- 2d Coulomb gaz is a special case of Normal 2-matrices

Complex matrix model

Choose $V_2 = \bar{V}_1$ and $\Gamma = \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid y = \bar{x}\}^N$ $\Lambda_1 = \Lambda\ ,\ \ \Lambda_2 = \bar{\Lambda}_1 = \bar{\Lambda}$

Measure $$d\mu(\Lambda_1,\Lambda_2) \ = |\Delta(\Lambda)|^2 \det e^{cN\lambda_i\bar{\lambda}_j} \prod_{i=1}^{N} e^{-N(V_1(\lambda_i)+V_2(\bar{\lambda}_i))}\ d^2\lambda_i$$

$$Z = N! \int_\Gamma \prod_{i<j} |\lambda_i - \lambda_j|^2 \ \prod_{i=1}^{N} e^{-N(2\Re(V_1(\lambda_i))-c|\lambda_i|^2)}\ d^2\lambda_i$$

- **Normal matrices** with homology class: $\Gamma = \{\gamma \times \bar{\gamma}\}^N$

  Or more generally with linear combinations $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \tilde{\gamma}_j)^N$

  Case $\tilde{\gamma}_i = \bar{\gamma}_i$ $\Gamma = (\sum_{i,j} c_{i,j}\gamma_i \times \bar{\gamma}_j)^N$ $\bar{c}_{i,j} = c_{j,i}$

- This is the 2-dimensional complex Coulomb gaz

- 2d Coulomb gaz is a special case of Normal 2-matrices

Limit shape

Limit shape

Limit density 1-matrix

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x - \lambda_i) \quad \in \mathfrak{M}(\gamma)$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x - \tilde{x}| \, d\nu(x) d\nu(\tilde{x})$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x - \tilde{x}| d\nu(x) d\nu(\tilde{x})$
- **Thm**

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\quad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x-\tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm**

$$\exists \sup_\gamma \inf_{\nu \in \mathfrak{M}(\gamma)} \mathcal{S}(\nu)$$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \int\int_{\gamma^2}' \ln|x - \tilde{x}| d\nu(x) d\nu(\tilde{x})$
- **Thm**

$\exists \sup_\gamma \inf_{\nu \in \mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint_{\gamma^2}' \ln |x - \tilde{x}| d\nu(x) d\nu(\tilde{x})$
- **Thm**

$\exists \sup_\gamma \inf_{\nu \in \mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

$\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x - \lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x - \tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm**

$\exists \sup_\gamma \inf_{\nu \in \mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

$\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

$$\forall\ x \in \operatorname{supp}\bar{\rho} \quad \Re\left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x - \tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x} \right) = \Re C$$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \mathfrak{R} V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x-\tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm**

$\exists \sup_\gamma \inf_{\nu \in \mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

$\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

$$\forall\ x \in \operatorname{supp}\bar{\rho} \quad \mathfrak{R}\left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x-\tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) = \mathfrak{R} C$$

$$\forall\ x \in \bar{\gamma} \backslash \operatorname{supp}\bar{\rho} \quad \left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x-\tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) \in C + \mathbb{R}_+$$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x-\tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm**

$\exists \ \sup_\gamma \ \inf_{\nu \in \mathfrak{M}(\gamma)} \ \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

$\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

$$\forall\ x \in \operatorname{supp}\bar{\rho} \quad \Re\left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x-\tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) = \Re C$$

$$\forall\ x \in \bar{\gamma}\backslash\operatorname{supp}\bar{\rho} \quad \left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x-\tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) \in C + \mathbb{R}_+$$

- Remark: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}dx$ is algebraic $\qquad \deg P = \deg V' - 1$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x - \lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x - \tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm**

$\exists \sup_\gamma \inf_{\nu \in \mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

$\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

$$\forall\, x \in \operatorname{supp} \bar{\rho} \quad \Re\left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x - \tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) = \Re C$$

$$\forall\, x \in \bar{\gamma} \backslash \operatorname{supp} \bar{\rho} \quad \left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x - \tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) \in C + \mathbb{R}_+$$

- Remark: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}\, dx$ is algebraic $\qquad \deg P = \deg V' - 1$
- Short-cut Method: Look for an algebraic solution of Euler-Lagrange → find $P(x)$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint_{\gamma^2}' \ln|x-\tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm** *[V. Baillard, B.E., E. Maurel-Segala, to be published soon, adaptation of potential theory]*

$\exists \sup_\gamma \inf_{\nu\in\mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar{\gamma}, \bar{\rho}$ the extremal arc and measure

$\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

$$\forall\, x \in \operatorname{supp}\bar{\rho} \quad \Re\left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x-\tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) = \Re C$$

$$\forall\, x \in \bar{\gamma}\backslash\operatorname{supp}\bar{\rho} \quad \left(V(x) - 2\int_{\operatorname{supp}\bar{\rho}} \ln(x-\tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) \in C + \mathbb{R}_+$$

- Remark: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}dx$ is algebraic $\qquad \deg P = \deg V' - 1$
- Short-cut Method: Look for an algebraic solution of Euler-Lagrange → find $P(x)$

Limit density 1-matrix

- Diagonal Matrix $\Lambda \in \Gamma = \gamma^N$
- Measure : $d\mu(\Lambda) = e^{-N^2 S(\Lambda)} \prod_i d\lambda_i$ $\qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) - \frac{2}{N^2}\sum_{i<j}\ln(\lambda_i - \lambda_j)$
- **Limit :** $N \to \infty$ replace Λ by its spectral density $\rho_\lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \quad \in \mathfrak{M}(\gamma)$
- Limit shape = minimum of energy $\mathcal{S}(\nu) = \int_\gamma \Re V(x) d\nu(x) - \iint'_{\gamma^2} \ln|x-\tilde{x}|\, d\nu(x) d\nu(\tilde{x})$
- **Thm** [V. Baillard, B.E., E. Maurel-Segala, to be published soon, adaptation of potential theory]

$\exists \sup_\gamma \inf_{\nu\in\mathfrak{M}(\gamma)} \mathcal{S}(\nu)$ Let $\bar\gamma, \bar\rho$ the extremal arc and measure

$\bar\gamma, \bar\rho$ = unique solution of Euler Lagrange equations:

$$\forall\ x \in \operatorname{supp}\bar\rho \quad \Re\left(V(x) - 2\int_{\operatorname{supp}\bar\rho} \ln(x-\tilde{x}) d\bar\rho(\tilde{x}) d\tilde{x}\right) = \Re C$$

$$\forall\ x \in \bar\gamma \backslash \operatorname{supp}\bar\rho \quad \left(V(x) - 2\int_{\operatorname{supp}\bar\rho} \ln(x-\tilde{x}) d\bar\rho(\tilde{x}) d\tilde{x}\right) \in C + \mathbb{R}_+$$

- Remark: $\bar\rho(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}\, dx$ is algebraic $\qquad \deg P = \deg V' - 1$
- Short-cut Method: Look for an algebraic solution of Euler-Lagrange → find $P(x)$

Limit density 1-matrix

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Ansatz: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ algebraic

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Ansatz: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ algebraic

→ find $P(x)$ such that $\bar{\rho}(x)dx > 0$

$$\forall\ x \in \mathrm{supp}_{\bar{\rho}}\ \Re\left(V(x) - 2\int_{\mathrm{supp}_{\bar{\rho}}} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = C$$

$$\forall\ x \in \bar{\gamma}\backslash\mathrm{supp}_{\bar{\rho}}\ \Re\left(V(x) - 2\int_{\mathrm{supp}_{\bar{\rho}}} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \geq C$$

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Ansatz: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ algebraic

→ find $P(x)$ such that $\bar{\rho}(x)dx > 0$

$$\forall\ x \in \text{supp}_{\bar{\rho}}\ \Re\left(V(x) - 2\int_{\text{supp}_{\bar{\rho}}} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = C$$

$$\forall\ x \in \bar{\gamma}\backslash\text{supp}_{\bar{\rho}}\ \Re\left(V(x) - 2\int_{\text{supp}_{\bar{\rho}}} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \geq C$$

- **Remark:** Stieltjes transform (resolvent) $\bar{W}(x) = \int_{\text{supp}\,\bar{\rho}} \frac{\bar{\rho}(x')dx'}{x - x'}$

satisfies an algebraic equation : $E(x, \bar{W}(x)) = 0$

$$\bar{W}(x)^2 - V'(x)\bar{W}(x) + P(x) = 0$$

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Ansatz: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ algebraic

→ find $P(x)$ such that $\bar{\rho}(x)dx > 0$

$$\forall\ x \in \text{supp}_{\bar{\rho}}\, \Re\left(V(x) - 2\int_{\text{supp}_{\bar{\rho}}} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = C$$

$$\forall\ x \in \bar{\gamma}\backslash\text{supp}_{\bar{\rho}}\, \Re\left(V(x) - 2\int_{\text{supp}_{\bar{\rho}}} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \geq C$$

- **Remark:** Stieltjes transform (resolvent) $\bar{W}(x) = \int_{\text{supp}\,\bar{\rho}} \frac{\bar{\rho}(x')dx'}{x-x'}$ $\quad \bar{W}(x) \underset{x\to\infty}{\sim} \frac{1}{x}$

satisfies an algebraic equation : $E(x, \bar{W}(x)) = 0$

$$\bar{W}(x)^2 - V'(x)\bar{W}(x) + P(x) = 0$$

$$\Longrightarrow \quad P(x) \underset{x\to\infty}{\sim} V'(x)/x \qquad \deg P = \deg V' - 1$$

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Ansatz: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ algebraic

→ find $P(x)$ such that $\bar{\rho}(x)dx > 0$

$$\forall\ x \in \mathrm{supp}_{\bar{\rho}}\, \Re\left(V(x) - 2\int_{\mathrm{supp}_{\bar{\rho}}} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = C$$

$$\forall\ x \in \bar{\gamma}\backslash\mathrm{supp}_{\bar{\rho}}\, \Re\left(V(x) - 2\int_{\mathrm{supp}_{\bar{\rho}}} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \geq C$$

- **Remark:** Stieltjes transform (resolvent) $\bar{W}(x) = \int_{\mathrm{supp}\,\bar{\rho}} \frac{\bar{\rho}(x')dx'}{x - x'}$ $\qquad \bar{W}(x) \underset{x\to\infty}{\sim} \frac{1}{x}$

satisfies an algebraic equation : $E(x, \bar{W}(x)) = 0$

$$\bar{W}(x)^2 - V'(x)\bar{W}(x) + P(x) = 0$$

$\Longrightarrow \quad P(x) \underset{x\to\infty}{\sim} V'(x)/x \qquad \deg P = \deg V' - 1 \qquad$ look for $\deg V' - 1$ unknown coefficients

Limit density 1-matrix

- Limit shape = minimum of energy $\bar{\gamma}, \bar{\rho}$ = unique solution of Euler Lagrange equations:

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Ansatz: $\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ algebraic

→ find $P(x)$ such that $\bar{\rho}(x)dx > 0$

$$\forall\ x \in \mathrm{supp}_{\bar{\rho}}\, \Re\left(V(x) - 2\int_{\mathrm{supp}_{\bar{\rho}}} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = C$$

$$\forall\ x \in \bar{\gamma}\backslash\mathrm{supp}_{\bar{\rho}}\, \Re\left(V(x) - 2\int_{\mathrm{supp}_{\bar{\rho}}} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \geq C$$

- **Remark:** Stieltjes transform (resolvent) $\bar{W}(x) = \int_{\mathrm{supp}\,\bar{\rho}} \frac{\bar{\rho}(x')dx'}{x-x'}$ $\bar{W}(x) \underset{x\to\infty}{\sim} \frac{1}{x}$

satisfies an algebraic equation : $E(x, \bar{W}(x)) = 0$

$$\bar{W}(x)^2 - V'(x)\bar{W}(x) + P(x) = 0$$

$\Longrightarrow$ $P(x) \underset{x\to\infty}{\sim} V'(x)/x$ $\deg P = \deg V' - 1$ look for $\deg V' - 1$ unknown coefficients

Example : Gaussian

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

$$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

$$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$$

$\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange ⟶ $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\Longrightarrow \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\Longrightarrow \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange ⟶ $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\Longrightarrow \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\Longrightarrow$ $\operatorname{supp}\bar{\rho} = [-2,2]$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange ⟶ $P(x)$

$$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$$

$\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\implies \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\implies$ $\operatorname{supp}\bar{\rho} = [-2,2]$

Euler Lagrange:

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange ➡ $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\Longrightarrow \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\Longrightarrow$ $\operatorname{supp}\bar{\rho} = [-2,2]$

Euler Lagrange: $V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x} = 1 + \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left(\frac{1}{2}(x + \sqrt{x^2 - 4})\right)$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\implies \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\implies \operatorname{supp}\bar{\rho} = [-2,2]$

Euler Lagrange: $V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x} = 1 + \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left(\frac{1}{2}(x + \sqrt{x^2 - 4})\right)$

$$\forall\ x \in [-2,2] \quad \Re\left(V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = 1$$

$$\forall\ x \in \mathbb{R}\backslash[-2,2] \quad \left(V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \in 1 + \mathbb{R}_+$$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\implies \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\implies \operatorname{supp}\bar{\rho} = [-2,2]$

Euler Lagrange: $$V(x) - 2\int_{-2}^{2} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x} = 1 + \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left(\frac{1}{2}(x+\sqrt{x^2-4})\right)$$

$$\forall\ x \in [-2,2] \quad \Re\left(V(x) - 2\int_{-2}^{2} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = 1$$

$$\forall\ x \in \mathbb{R}\backslash[-2,2] \quad \left(V(x) - 2\int_{-2}^{2} \ln(x-\tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \in 1 + \mathbb{R}_+$$

$\implies \bar{\gamma}, \bar{\rho}$ = solution of Euler Lagrange equations

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange ⟶ $P(x)$

$\bar\rho(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\implies \quad \bar\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp}\bar\rho$ where $\bar\rho(x)dx$ is a positive real measure $\implies \operatorname{supp}\bar\rho = [-2,2]$

Euler Lagrange: $V(x) - 2\int_{-2}^{2} \ln(x-\tilde x)d\bar\rho(\tilde x)d\tilde x = 1 + \frac{1}{2}x\sqrt{x^2-4} - 2\ln\left(\frac{1}{2}(x+\sqrt{x^2-4})\right)$

$$\forall\ x \in [-2,2] \quad \Re\left(V(x) - 2\int_{-2}^{2} \ln(x-\tilde x)d\bar\rho(\tilde x)d\tilde x\right) = 1$$

$$\forall\ x \in \mathbb{R}\backslash[-2,2] \quad \left(V(x) - 2\int_{-2}^{2} \ln(x-\tilde x)d\bar\rho(\tilde x)d\tilde x\right) \in 1 + \mathbb{R}_+$$

$\implies \bar\gamma, \bar\rho$ = solution of Euler Lagrange equations = unique

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange ⟶ $P(x)$

$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2}$ $\deg P = \deg V' - 1 = 0$ $P = 1$ (mass =1)

$$\Longrightarrow \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp} \bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\Longrightarrow$ $\operatorname{supp} \bar{\rho} = [-2,2]$

Euler Lagrange: $V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x} = 1 + \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left(\frac{1}{2}(x + \sqrt{x^2 - 4})\right)$

$$\forall\ x \in [-2,2] \quad \Re\left(V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) = 1$$

$$\forall\ x \in \mathbb{R}\backslash[-2,2] \quad \left(V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x}) d\bar{\rho}(\tilde{x}) d\tilde{x}\right) \in 1 + \mathbb{R}_+$$

$\Longrightarrow \bar{\gamma}, \bar{\rho}$ = solution of Euler Lagrange equations = unique

$$\bar{\gamma} = \mathbb{R} \qquad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

Example : Gaussian

- **Potential** $V(x) = \frac{1}{2}x^2$
- **Short-cut Method:** Look for an **algebraic** solution of Euler-Lagrange → $P(x)$

$$\bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2} \qquad \deg P = \deg V' - 1 = 0 \qquad P = 1 \quad \text{(mass =1)}$$

$$\implies \quad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

find $\operatorname{supp} \bar{\rho}$ where $\bar{\rho}(x)dx$ is a positive real measure $\implies \operatorname{supp} \bar{\rho} = [-2,2]$

Euler Lagrange: $$V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x} = 1 + \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left(\frac{1}{2}(x + \sqrt{x^2 - 4})\right)$$

$$\forall\, x \in [-2,2] \quad \Re\left(V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) = 1$$

$$\forall\, x \in \mathbb{R}\backslash[-2,2] \quad \left(V(x) - 2\int_{-2}^{2} \ln(x - \tilde{x})d\bar{\rho}(\tilde{x})d\tilde{x}\right) \in 1 + \mathbb{R}_+$$

$\implies \bar{\gamma}, \bar{\rho}$ = solution of Euler Lagrange equations = unique

$$\bar{\gamma} = \mathbb{R} \qquad \bar{\rho}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

Convergence

Convergence

Convergence

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

  then $\bar{\gamma} = \mathbb{R}$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} = \text{Argmin}\, \mathcal{S}$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} = \text{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \quad \in \mathfrak{M}(\mathbb{R})$

converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} \; = \mathrm{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

[Guionnet-Zeitouni 2000, BenArous,…] (also large deviations)

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} = \text{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

[Guionnet-Zeitouni 2000, BenArous,…] (also large deviations)

- **Normal Complex Case ?** $V(x) \in \mathbb{C}[x]$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

  then $\bar{\gamma} = \mathbb{R}$

  empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

  converges almost surely $\rho_\Lambda \xrightarrow{a.s} \bar{\rho} = \text{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

  [Guionnet-Zeitouni 2000, BenArous,…] (also large deviations)

- **Normal Complex Case ?** $V(x) \in \mathbb{C}[x]$

  empirical spectral density $\Lambda \in \bar{\gamma}^N$ $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\bar{\gamma})$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \in \mathfrak{M}(\mathbb{R})$

converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} = \text{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

[Guionnet-Zeitouni 2000, BenArous,…] (also large deviations)

- **Normal Complex Case ?** $V(x) \in \mathbb{C}[x]$

empirical spectral density $\Lambda \in \bar{\gamma}^N$ $\rho_\Lambda(x) = \frac{1}{N}\sum_i \delta(x-\lambda_i) \in \mathfrak{M}(\bar{\gamma})$

converges ? $\rho_\Lambda \overset{a.s}{\to} \bar{\rho}$

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

  then $\bar{\gamma} = \mathbb{R}$

  empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

  converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} = \text{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

  [Guionnet-Zeitouni 2000, BenArous,…] (also large deviations)

- **Normal Complex Case ?** $V(x) \in \mathbb{C}[x]$

  empirical spectral density $\Lambda \in \bar{\gamma}^N$ $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\bar{\gamma})$

  converges ? $\rho_\Lambda \overset{a.s}{\to} \bar{\rho}$

- **Partial proofs**

  Riemann Hilbert method, steepest descent method *[Deift&co, Bleher Its,…]*

Convergence

- **Thm** In the Real case $V(x) \in \mathbb{R}[x]$ $\inf_{\mathbb{R}} V > -\infty$ $\deg V \geq 2$

then $\bar{\gamma} = \mathbb{R}$

empirical spectral density $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\mathbb{R})$

converges almost surely $\rho_\Lambda \overset{a.s}{\to} \bar{\rho} = \text{Argmin}\, \mathcal{S}$ (also converges weakly and in proba)

[Guionnet-Zeitouni 2000, BenArous,…] (also large deviations)

- **Normal Complex Case ?** $V(x) \in \mathbb{C}[x]$

empirical spectral density $\Lambda \in \bar{\gamma}^N$ $\rho_\Lambda(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i) \in \mathfrak{M}(\bar{\gamma})$

converges ? $\rho_\Lambda \overset{a.s}{\to} \bar{\rho}$

- **Partial proofs**

Riemann Hilbert method, steepest descent method *[Deift&co, Bleher Its,…]*

2 - matrix model

Complex - matrix model

2 - matrix model

Complex - matrix model

Limit densities 2-matrix

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde{\gamma}^N} d\mu(\Lambda, \tilde{\Lambda}) = \int_{\gamma^N \times \tilde{\gamma}^N} e^{-N^2 S(\Lambda, \tilde{\Lambda})} \prod_i d\lambda_i \prod_i d\tilde{\lambda}_i \qquad I(\Lambda, \tilde{\Lambda}) = \frac{\det e^{Nc\lambda_i \tilde{\lambda}_j}}{\prod_{i<j} (\lambda_i - \lambda_j)(\tilde{\lambda}_i - \tilde{\lambda}_j)}$$

$$S(\Lambda, \tilde{\Lambda}) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N} \sum_{i=1}^{N} \tilde{V}(\tilde{\lambda}_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j) - \frac{2}{N^2} \sum_{i<j} \ln(\tilde{\lambda}_i - \tilde{\lambda}_j) - \log I(\Lambda, \tilde{\Lambda})$$

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde{\gamma}^N} d\mu(\Lambda, \tilde{\Lambda}) = \int_{\gamma^N \times \tilde{\gamma}^N} e^{-N^2 S(\Lambda, \tilde{\Lambda})} \prod_i d\lambda_i \prod_i d\tilde{\lambda}_i \qquad I(\Lambda, \tilde{\Lambda}) = \frac{\det e^{Nc\lambda_i \tilde{\lambda}_j}}{\prod_{i<j} (\lambda_i - \lambda_j)(\tilde{\lambda}_i - \tilde{\lambda}_j)}$$

$$S(\Lambda, \tilde{\Lambda}) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N} \sum_{i=1}^{N} \tilde{V}(\tilde{\lambda}_i) - \frac{2}{N^2} \sum_{i<j} \ln(\lambda_i - \lambda_j) - \frac{2}{N^2} \sum_{i<j} \ln(\tilde{\lambda}_i - \tilde{\lambda}_j) - \log I(\Lambda, \tilde{\Lambda})$$

- Limit shape = minimum of energy

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde\gamma^N} d\mu(\Lambda, \tilde\Lambda) = \int_{\gamma^N \times \tilde\gamma^N} e^{-N^2 S(\Lambda,\tilde\Lambda)} \prod_i d\lambda_i \prod_i d\tilde\lambda_i \qquad I(\Lambda,\tilde\Lambda) = \frac{\det e^{Nc\lambda_i\tilde\lambda_j}}{\prod_{i<j}(\lambda_i-\lambda_j)(\tilde\lambda_i-\tilde\lambda_j)}$$

$$S(\Lambda,\tilde\Lambda) = \frac{1}{N}\sum_{i=1}^N V(\lambda_i) + \frac{1}{N}\sum_{i=1}^N \tilde V(\tilde\lambda_i) - \frac{2}{N^2}\sum_{i<j}\ln(\lambda_i-\lambda_j) - \frac{2}{N^2}\sum_{i<j}\ln(\tilde\lambda_i-\tilde\lambda_j) - \log I(\Lambda,\tilde\Lambda)$$

- Limit shape = minimum of energy $\quad \nu \in \mathfrak{M}(\gamma),\ \tilde\nu \in \mathfrak{M}(\tilde\gamma)$

$$\mathcal{S}(\nu,\tilde\nu) = \int_\gamma \Re V(x)d\nu(x) + \int_{\tilde\gamma} \Re \tilde V(x) d\tilde\nu(x) - \frac{1}{2}\iint'_{\gamma^2} \ln|x-x'|\, d\nu(x)d\nu(x') - \frac{1}{2}\iint'_{\tilde\gamma^2} \ln|y-y'|\, d\tilde\nu(y)d\tilde\nu(y')$$
$$-\log \mathcal{I}(\nu,\tilde\nu)$$

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde{\gamma}^N} d\mu(\Lambda, \tilde{\Lambda}) = \int_{\gamma^N \times \tilde{\gamma}^N} e^{-N^2 S(\Lambda, \tilde{\Lambda})} \prod_i d\lambda_i \prod_i d\tilde{\lambda}_i \qquad I(\Lambda, \tilde{\Lambda}) = \frac{\det e^{Nc\lambda_i \tilde{\lambda}_j}}{\prod_{i<j}(\lambda_i - \lambda_j)(\tilde{\lambda}_i - \tilde{\lambda}_j)}$$

$$S(\Lambda, \tilde{\Lambda}) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N}\sum_{i=1}^{N} \tilde{V}(\tilde{\lambda}_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j) - \frac{2}{N^2}\sum_{i<j} \ln(\tilde{\lambda}_i - \tilde{\lambda}_j) - \log I(\Lambda, \tilde{\Lambda})$$

- Limit shape = minimum of energy $\quad \nu \in \mathfrak{M}(\gamma),\ \tilde{\nu} \in \mathfrak{M}(\tilde{\gamma})$

$$\mathcal{S}(\nu, \tilde{\nu}) = \int_\gamma \Re V(x) d\nu(x) + \int_{\tilde{\gamma}} \Re \tilde{V}(x) d\tilde{\nu}(x) - \frac{1}{2} \int\int_{\gamma^2}' \ln|x - x'| d\nu(x) d\nu(x') - \frac{1}{2} \int\int_{\tilde{\gamma}^2}' \ln|y - y'| d\tilde{\nu}(y) d\tilde{\nu}(y')$$
$$-\log \mathcal{I}(\nu, \tilde{\nu})$$

Matytsin relation: $\dfrac{d}{dx}\dfrac{\partial}{\partial d\nu(x)} \mathcal{I}(\nu, \tilde{\nu}) \circ \dfrac{d}{dy}\dfrac{\partial}{\partial d\tilde{\nu}(y)} \mathcal{I}(\nu, \tilde{\nu}) = \mathrm{Id}$

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde\gamma^N} d\mu(\Lambda, \tilde\Lambda) = \int_{\gamma^N \times \tilde\gamma^N} e^{-N^2 S(\Lambda,\tilde\Lambda)} \prod_i d\lambda_i \prod_i d\tilde\lambda_i \qquad I(\Lambda,\tilde\Lambda) = \frac{\det e^{Nc\lambda_i\tilde\lambda_j}}{\prod_{i<j}(\lambda_i-\lambda_j)(\tilde\lambda_i-\tilde\lambda_j)}$$

$$S(\Lambda,\tilde\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N}\sum_{i=1}^{N} \tilde V(\tilde\lambda_i) - \frac{2}{N^2}\sum_{i<j}\ln(\lambda_i-\lambda_j) - \frac{2}{N^2}\sum_{i<j}\ln(\tilde\lambda_i-\tilde\lambda_j) - \log I(\Lambda,\tilde\Lambda)$$

- Limit shape = minimum of energy $\quad \nu \in \mathfrak{M}(\gamma),\ \tilde\nu \in \mathfrak{M}(\tilde\gamma)$

$$\mathcal{S}(\nu,\tilde\nu) = \int_\gamma \Re V(x)d\nu(x) + \int_{\tilde\gamma} \Re \tilde V(x) d\tilde\nu(x) - \frac{1}{2}\iint'_{\gamma^2} \ln|x-x'|\,d\nu(x)d\nu(x') - \frac{1}{2}\iint'_{\tilde\gamma^2} \ln|y-y'|\,d\tilde\nu(y)d\tilde\nu(y')$$
$$-\log \mathcal{I}(\nu,\tilde\nu)$$

Matytsin relation: $\frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu,\tilde\nu) \circ \frac{d}{dy}\frac{\partial}{\partial d\tilde\nu(y)}\mathcal{I}(\nu,\tilde\nu) = \mathrm{Id}$

$\bar\gamma, \bar\nu, \bar{\tilde\gamma}, \bar{\tilde\nu}$ = unique solution of Euler Lagrange equations:

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \int_\gamma \frac{d\nu(x')}{x-x'} \qquad Y(x) = \frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu,\tilde\nu)$$

$$X \circ Y = \mathrm{Id}$$

$$\tilde V'(y) - \tilde W(y) - cX(y) = 0 \qquad \tilde W(y) = \int_{\tilde\gamma} \frac{d\tilde\nu(y')}{y-y'} \qquad X(y) = \frac{d}{dy}\frac{\partial}{\partial d\tilde\nu(y)}\mathcal{I}(\nu,\tilde\nu)$$

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde\gamma^N} d\mu(\Lambda, \tilde\Lambda) = \int_{\gamma^N \times \tilde\gamma^N} e^{-N^2 S(\Lambda,\tilde\Lambda)} \prod_i d\lambda_i \prod_i d\tilde\lambda_i \qquad I(\Lambda, \tilde\Lambda) = \frac{\det e^{Nc\lambda_i\tilde\lambda_j}}{\prod_{i<j}(\lambda_i - \lambda_j)(\tilde\lambda_i - \tilde\lambda_j)}$$

$$S(\Lambda, \tilde\Lambda) = \frac{1}{N}\sum_{i=1}^N V(\lambda_i) + \frac{1}{N}\sum_{i=1}^N \tilde V(\tilde\lambda_i) - \frac{2}{N^2}\sum_{i<j}\ln(\lambda_i - \lambda_j) - \frac{2}{N^2}\sum_{i<j}\ln(\tilde\lambda_i - \tilde\lambda_j) - \log I(\Lambda, \tilde\Lambda)$$

- Limit shape = minimum of energy $\quad \nu \in \mathfrak{M}(\gamma),\ \tilde\nu \in \mathfrak{M}(\tilde\gamma)$

$$\mathcal{S}(\nu, \tilde\nu) = \int_\gamma \Re V(x) d\nu(x) + \int_{\tilde\gamma} \Re \tilde V(x) d\tilde\nu(x) - \frac{1}{2}\iint'_{\gamma^2} \ln|x - x'|\, d\nu(x) d\nu(x') - \frac{1}{2}\iint'_{\tilde\gamma^2} \ln|y - y'|\, d\tilde\nu(y) d\tilde\nu(y')$$
$$-\log \mathcal{I}(\nu, \tilde\nu)$$

Matytsin relation: $\displaystyle \frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu,\tilde\nu) \circ \frac{d}{dy}\frac{\partial}{\partial d\tilde\nu(y)}\mathcal{I}(\nu,\tilde\nu) = \mathrm{Id}$

$\bar\gamma, \bar\nu, \bar{\tilde\gamma}, \bar{\tilde\nu}$ = unique solution of Euler Lagrange equations:

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \int_\gamma \frac{d\nu(x')}{x - x'} \qquad Y(x) = \frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu, \tilde\nu)$$

$$X \circ Y = \mathrm{Id}$$

$$\tilde V'(y) - \tilde W(y) - cX(y) = 0 \qquad \tilde W(y) = \int_{\tilde\gamma} \frac{d\tilde\nu(y')}{y - y'} \qquad X(y) = \frac{d}{dy}\frac{\partial}{\partial d\tilde\nu(y)}\mathcal{I}(\nu, \tilde\nu)$$

- Remark: $\quad W(x), \tilde W(y), Y(x), X(y)\quad$ are algebraic $\quad E(x, Y(x)) = 0 \qquad E(X(y), y) = 0$

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde{\gamma}^N} d\mu(\Lambda, \tilde{\Lambda}) = \int_{\gamma^N \times \tilde{\gamma}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d\lambda_i \prod_i d\tilde{\lambda}_i \qquad I(\Lambda, \tilde{\Lambda}) = \frac{\det e^{Nc\lambda_i\tilde{\lambda}_j}}{\prod_{i<j}(\lambda_i - \lambda_j)(\tilde{\lambda}_i - \tilde{\lambda}_j)}$$

$$S(\Lambda, \tilde{\Lambda}) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N}\sum_{i=1}^{N} \tilde{V}(\tilde{\lambda}_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j) - \frac{2}{N^2}\sum_{i<j} \ln(\tilde{\lambda}_i - \tilde{\lambda}_j) - \log I(\Lambda, \tilde{\Lambda})$$

- Limit shape = minimum of energy $\quad \nu \in \mathfrak{M}(\gamma),\ \tilde{\nu} \in \mathfrak{M}(\tilde{\gamma})$

$$\mathcal{S}(\nu, \tilde{\nu}) = \int_\gamma \Re V(x) d\nu(x) + \int_{\tilde{\gamma}} \Re \tilde{V}(x) d\tilde{\nu}(x) - \frac{1}{2}\iint'_{\gamma^2} \ln|x - x'|\, d\nu(x) d\nu(x') - \frac{1}{2}\iint'_{\tilde{\gamma}^2} \ln|y - y'|\, d\tilde{\nu}(y) d\tilde{\nu}(y')$$
$$-\log \mathcal{I}(\nu, \tilde{\nu})$$

Matytsin relation: $\frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu, \tilde{\nu}) \circ \frac{d}{dy}\frac{\partial}{\partial d\tilde{\nu}(y)}\mathcal{I}(\nu, \tilde{\nu}) = \mathrm{Id}$

$\bar{\gamma}, \bar{\nu}, \bar{\tilde{\gamma}}, \bar{\tilde{\nu}}$ = unique solution of Euler Lagrange equations:

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \int_\gamma \frac{d\nu(x')}{x - x'} \qquad Y(x) = \frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu, \tilde{\nu})$$

$$X \circ Y = \mathrm{Id}$$

$$\tilde{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \int_{\tilde{\gamma}} \frac{d\tilde{\nu}(y')}{y - y'} \qquad X(y) = \frac{d}{dy}\frac{\partial}{\partial d\tilde{\nu}(y)}\mathcal{I}(\nu, \tilde{\nu})$$

- Remark: $W(x), \tilde{W}(y), Y(x), X(y)$ are algebraic $\quad E(x, Y(x)) = 0 \quad E(X(y), y) = 0$
- Short-cut Method: Look for an algebraic solution of Euler-Lagrange → find $E(x, y)$

Limit densities 2-matrix

$$Z = \int_{\gamma^N \times \tilde{\gamma}^N} d\mu(\Lambda, \tilde{\Lambda}) = \int_{\gamma^N \times \tilde{\gamma}^N} e^{-N^2 S(\Lambda, \tilde{\Lambda})} \prod_i d\lambda_i \prod_i d\tilde{\lambda}_i \qquad I(\Lambda, \tilde{\Lambda}) = \frac{\det e^{Nc\lambda_i \tilde{\lambda}_j}}{\prod_{i<j}(\lambda_i - \lambda_j)(\tilde{\lambda}_i - \tilde{\lambda}_j)}$$

$$S(\Lambda, \tilde{\Lambda}) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N}\sum_{i=1}^{N} \tilde{V}(\tilde{\lambda}_i) - \frac{2}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j) - \frac{2}{N^2}\sum_{i<j} \ln(\tilde{\lambda}_i - \tilde{\lambda}_j) - \log I(\Lambda, \tilde{\Lambda})$$

- Limit shape = minimum of energy $\quad \nu \in \mathfrak{M}(\gamma),\ \tilde{\nu} \in \mathfrak{M}(\tilde{\gamma})$

$$\mathcal{S}(\nu, \tilde{\nu}) = \int_\gamma \Re V(x) d\nu(x) + \int_{\tilde{\gamma}} \Re \tilde{V}(x) d\tilde{\nu}(x) - \frac{1}{2}\iint'_{\gamma^2} \ln|x - x'| d\nu(x) d\nu(x') - \frac{1}{2}\iint'_{\tilde{\gamma}^2} \ln|y - y'| d\tilde{\nu}(y) d\tilde{\nu}(y')$$
$$-\log \mathcal{I}(\nu, \tilde{\nu})$$

Matytsin relation: $\displaystyle \frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu, \tilde{\nu}) \circ \frac{d}{dy}\frac{\partial}{\partial d\tilde{\nu}(y)}\mathcal{I}(\nu, \tilde{\nu}) = \mathrm{Id}$

$\bar{\gamma}, \bar{\nu}, \bar{\tilde{\gamma}}, \bar{\tilde{\nu}}$ = unique solution of Euler Lagrange equations:

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \int_\gamma \frac{d\nu(x')}{x - x'} \qquad Y(x) = \frac{d}{dx}\frac{\partial}{\partial d\nu(x)}\mathcal{I}(\nu, \tilde{\nu})$$

$$X \circ Y = \mathrm{Id}$$

$$\tilde{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \int_{\tilde{\gamma}} \frac{d\tilde{\nu}(y')}{y - y'} \qquad X(y) = \frac{d}{dy}\frac{\partial}{\partial d\tilde{\nu}(y)}\mathcal{I}(\nu, \tilde{\nu})$$

- Remark: $W(x), \tilde{W}(y), Y(x), X(y)$ are algebraic $\quad E(x, Y(x)) = 0 \quad E(X(y), y) = 0$

- Short-cut Method: Look for an algebraic solution of Euler-Lagrange → find $E(x, y)$

→

Limit density Complex-matrix

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i$$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2} \sum_{i<j} \ln|\lambda_i - \lambda_j|$$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda, \tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2} \sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2} \int_{\mathrm{supp}\,\nu} \int_{\mathrm{supp}\,\nu}' \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i$$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2} \sum_{i<j} \ln|\lambda_i - \lambda_j|$$

$c < 0$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2} \int_{\mathrm{supp}\,\nu} \int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

$c < 0$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2)d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x)d\nu(x')$$

Euler-Lagrange equations

$$\Delta\nu = -c/\pi \quad \text{on } \mathrm{supp}\,\nu$$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda, \tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i$$

$$S(\Lambda) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2} \sum_{i<j} \ln|\lambda_i - \lambda_j|$$

$c < 0$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2} \int_{\mathrm{supp}\,\nu} \int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ $\nu =$ constant density on a Domain $\mathcal{D} = \mathrm{supp}\,\nu$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ $\qquad \nu$ = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary

$\partial\mathscr{D}$ = curve

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i$$

$$S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

$c < 0$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary $\partial\mathscr{D}$ = curve

$V'(x) - W(x) - cY(x) = 0$ $\qquad W(x) = \dfrac{1}{2i}\displaystyle\int_D \frac{1}{x - \bar{x}'}\, dx' \wedge d\bar{x}'$ $\qquad \bar{z} = Y(z)$ = Schwartz function

Stieltjes transform of constant density of the domain

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ $\quad\nu =$ constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary

$\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i$$

$$S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'} dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$X \circ Y =$ **Id**

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'} dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\quad \tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ $\quad$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$ $\quad$ Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'} dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$X \circ Y =$ **Id**

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'} dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

- **Short-cut Method:**

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2)d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x)d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$X \circ Y =$ **Id**

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$ Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$X \circ Y =$ **Id**

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

  find a polynomial $E(x, y)$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

$c < 0$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$ Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$X \circ Y =$ **Id**

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'}\, dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

  find a polynomial $E(x, y)$ Boundary $\partial\mathscr{D}$ of equation $E(z, \bar{z}) = 0$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$c < 0$

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'} dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$X \circ Y = \mathbf{Id}$

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'} dx' \wedge d\bar{x}'$$

$z = X(\bar{z})$ = reciprocal Schwartz function

Stieltjes transform of constant density of the domain

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

  find a polynomial $E(x, y)$ Boundary $\partial\mathscr{D}$ of equation $E(z, \bar{z}) = 0$

$$E(x, y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x, y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

Limit density Complex-matrix

Complex Matrix $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ $\tilde{\Lambda} := \bar{\Lambda}$ complex conjugate of Λ

$$Z = \int_{\mathbb{C}^N} e^{-N^2 S(\Lambda,\tilde{\Lambda})} \prod_i d^2\lambda_i \prod_i \qquad S(\Lambda) = \frac{1}{N}\sum_{i=1}^{N} V(\lambda_i) + \bar{V}(\bar{\lambda}_i) - c\,|\lambda_i|^2 - \frac{2}{N^2}\sum_{i<j} \ln|\lambda_i - \lambda_j|$$

$c < 0$

- Find density $\nu \in \mathfrak{M}(\mathbb{C})$ minimizing the energy

$$\mathcal{S}(\nu) = \int_{\mathrm{supp}\,\nu} (V(x) + \bar{V}(\bar{x}) - c\,|x|^2) d\nu(x) - \frac{1}{2}\int_{\mathrm{supp}\,\nu}\int'_{\mathrm{supp}\,\nu} \ln|x - x'|^2 d\nu(x) d\nu(x')$$

Euler-Lagrange equations

$\Delta\nu = -c/\pi$ on $\mathrm{supp}\,\nu$ ν = constant density on a Domain $\mathscr{D} = \mathrm{supp}\,\nu$

Boundary $\partial\mathscr{D}$ = curve

$$V'(x) - W(x) - cY(x) = 0 \qquad W(x) = \frac{1}{2i}\int_D \frac{1}{x - \bar{x}'} dx' \wedge d\bar{x}'$$

$$\bar{V}'(y) - \tilde{W}(y) - cX(y) = 0 \qquad \tilde{W}(y) = \frac{-1}{2i}\int_D \frac{1}{y - \bar{x}'} dx' \wedge d\bar{x}'$$

$\bar{z} = Y(z)$ = Schwartz function

$z = X(\bar{z})$ = reciprocal Schwartz function

$X \circ Y = \mathbf{Id}$

Stieltjes transform of constant density of the domain

- **Short-cut Method:** Look for an algebraic solution of Euler-Lagrange

find a polynomial $E(x, y)$ Boundary $\partial\mathscr{D}$ of equation $E(z, \bar{z}) = 0$

$$E(x, y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x, y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

Example quadratic

Example quadratic

$V(x) = a\frac{x^2}{2}$ $\bar{V}(y) = \bar{a}\frac{y^2}{2}$ $|a| < |c|$ $c < 0$

Example quadratic

$V(x) = a\frac{x^2}{2}$ $\bar{V}(y) = \bar{a}\frac{y^2}{2}$ $|a| < |c|$ $c < 0$

$E(x,y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x,y)$ $\deg P = (\deg V' - 1, \deg \bar{V}' - 1)$

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x, y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x, y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x, y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x,y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x,y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x,y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$

$$Y(x) = \frac{1}{2\bar{a}c}\left(x(c^2 + a\bar{a}) \pm \sqrt{x^2(c^2 - a\bar{a})^2 - 4\bar{a}(|a|^2 - c^2)}\right)$$

$$X(y) = \frac{1}{2ac}\left(y(c^2 + a\bar{a}) \pm \sqrt{y^2(c^2 - a\bar{a})^2 - 4a(|a|^2 - c^2)}\right)$$

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x,y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x,y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x,y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$

$$Y(x) = \frac{1}{2\bar{a}c}\left(x(c^2 + a\bar{a}) \pm \sqrt{x^2(c^2 - a\bar{a})^2 - 4\bar{a}(|a|^2 - c^2)}\right)$$
$$X(y) = \frac{1}{2ac}\left(y(c^2 + a\bar{a}) \pm \sqrt{y^2(c^2 - a\bar{a})^2 - 4a(|a|^2 - c^2)}\right)$$

Domain D of boundary ∂D

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x, y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x, y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x, y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$

$$Y(x) = \frac{1}{2\bar{a}c}\left(x(c^2 + a\bar{a}) \pm \sqrt{x^2(c^2 - a\bar{a})^2 - 4\bar{a}(|a|^2 - c^2)}\right)$$

$$X(y) = \frac{1}{2ac}\left(y(c^2 + a\bar{a}) \pm \sqrt{y^2(c^2 - a\bar{a})^2 - 4a(|a|^2 - c^2)}\right)$$

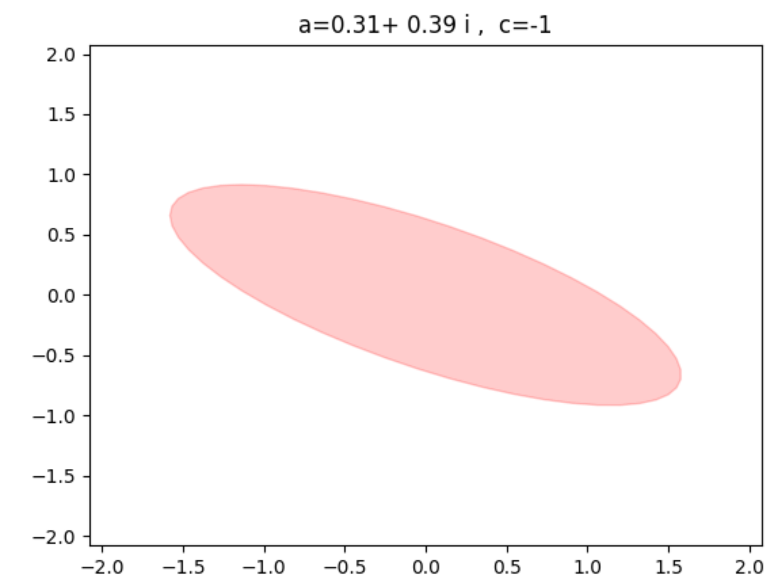
Domain D of boundary ∂D $\qquad (az - c\bar{z})(\bar{a}\bar{z} - cz) = -\frac{c^2 - |a|^2}{c}$

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x,y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x,y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x,y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$



$$Y(x) = \frac{1}{2\bar{a}c}\left(x(c^2 + a\bar{a}) \pm \sqrt{x^2(c^2 - a\bar{a})^2 - 4\bar{a}(|a|^2 - c^2)}\right)$$

$$X(y) = \frac{1}{2ac}\left(y(c^2 + a\bar{a}) \pm \sqrt{y^2(c^2 - a\bar{a})^2 - 4a(|a|^2 - c^2)}\right)$$

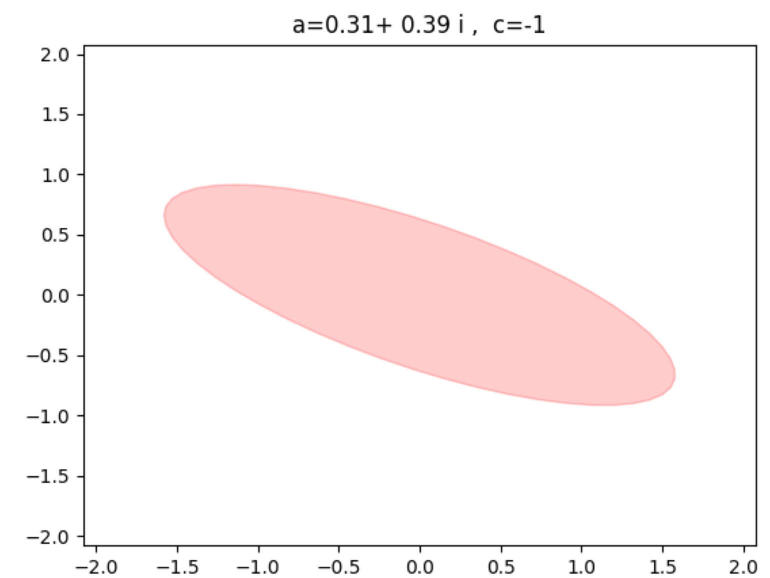
Domain D of boundary ∂D $\qquad (az - c\bar{z})(\bar{a}\bar{z} - cz) = -\frac{c^2 - |a|^2}{c}$ = Ellipse

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x,y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x,y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x,y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$



$$Y(x) = \frac{1}{2\bar{a}c}\left(x(c^2 + a\bar{a}) \pm \sqrt{x^2(c^2 - a\bar{a})^2 - 4\bar{a}(|a|^2 - c^2)}\right)$$

$$X(y) = \frac{1}{2ac}\left(y(c^2 + a\bar{a}) \pm \sqrt{y^2(c^2 - a\bar{a})^2 - 4a(|a|^2 - c^2)}\right)$$

Domain D of boundary ∂D $\qquad (az - c\bar{z})(\bar{a}\bar{z} - cz) = -\frac{c^2 - |a|^2}{c}$ = Ellipse

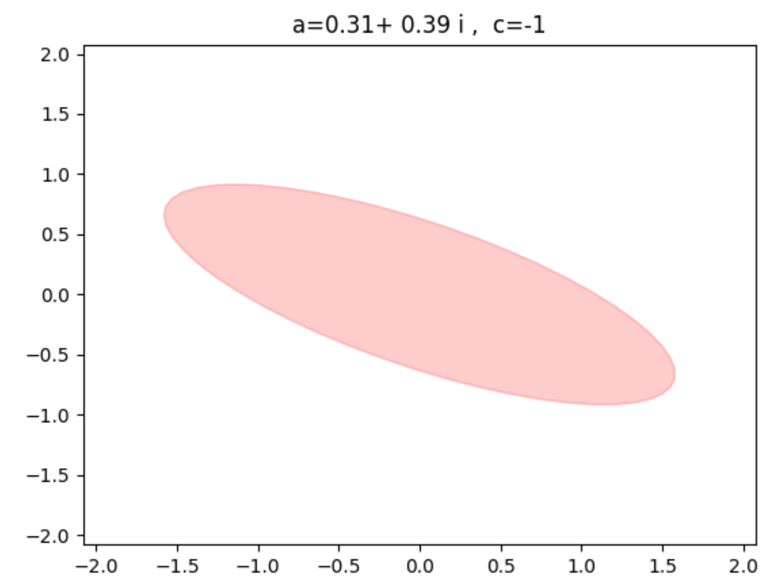
$$z = \sqrt{\frac{c^2 - |a|^2}{c}}\,\left(\bar{a}e^{i\theta} + ce^{-i\theta}\right)$$

Example quadratic

$$V(x) = a\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^2}{2} \qquad |a| < |c| \qquad c < 0$$

$$E(x,y) = (V'(x) - cy)(\bar{V}'(y) - cx) - P(x,y) \qquad \deg P = (\deg V' - 1, \deg \bar{V}' - 1)$$

$$E(x,y) = (ax - cy)(\bar{a}y - cx) + \frac{c^2 - a\bar{a}}{c}$$



$$Y(x) = \frac{1}{2\bar{a}c}\left(x(c^2 + a\bar{a}) \pm \sqrt{x^2(c^2 - a\bar{a})^2 - 4\bar{a}(|a|^2 - c^2)}\right)$$

$$X(y) = \frac{1}{2ac}\left(y(c^2 + a\bar{a}) \pm \sqrt{y^2(c^2 - a\bar{a})^2 - 4a(|a|^2 - c^2)}\right)$$

Domain D of boundary ∂D $\qquad (az - c\bar{z})(\bar{a}\bar{z} - cz) = -\dfrac{c^2 - |a|^2}{c}$ = Ellipse

$$z = \sqrt{\frac{c^2 - |a|^2}{c}}\,\left(\bar{a}e^{i\theta} + ce^{-i\theta}\right)$$

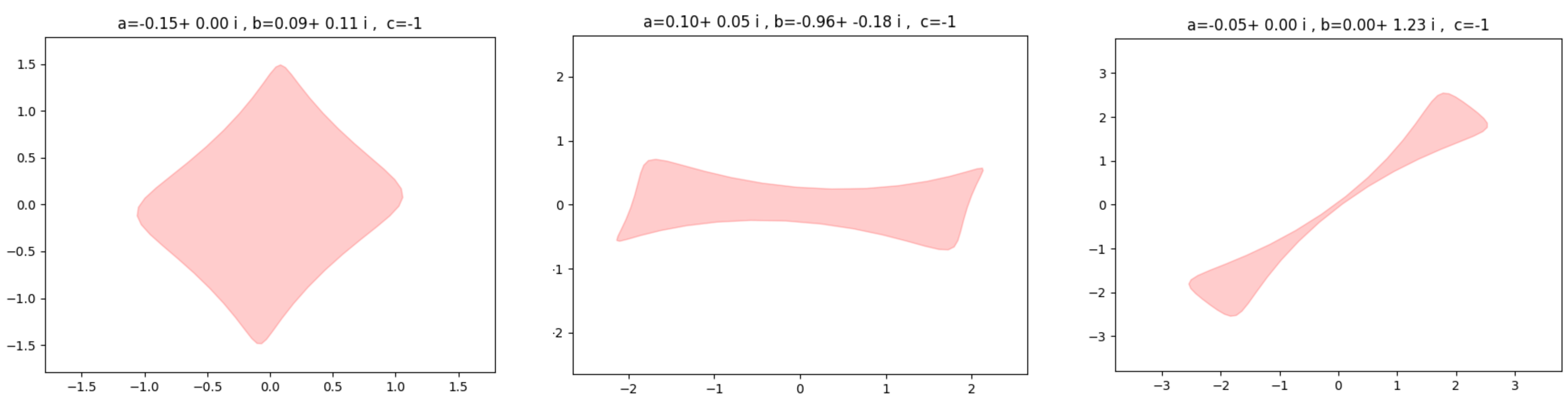
Example quartic

$$V(x) = a\frac{x^4}{4} + b\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^4}{4} + \bar{b}\frac{y^2}{2} \qquad \prod_i e^{-N(-c|\lambda_i|^2 + 2\Re V(\lambda_i))} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

$$E(x,y) = (ax^3 + bx - cy)(\bar{a}y^3 + \bar{b}y - cx) - \frac{1}{c}(a\bar{a}x^2y^2 + px^2 + \bar{p}y^2 + K) + c$$

Find $p \in \mathbb{C},\ K \in \mathbb{R}$ that minimise energy

Domain D of boundary ∂D $E(z,\bar{z}) = 0$



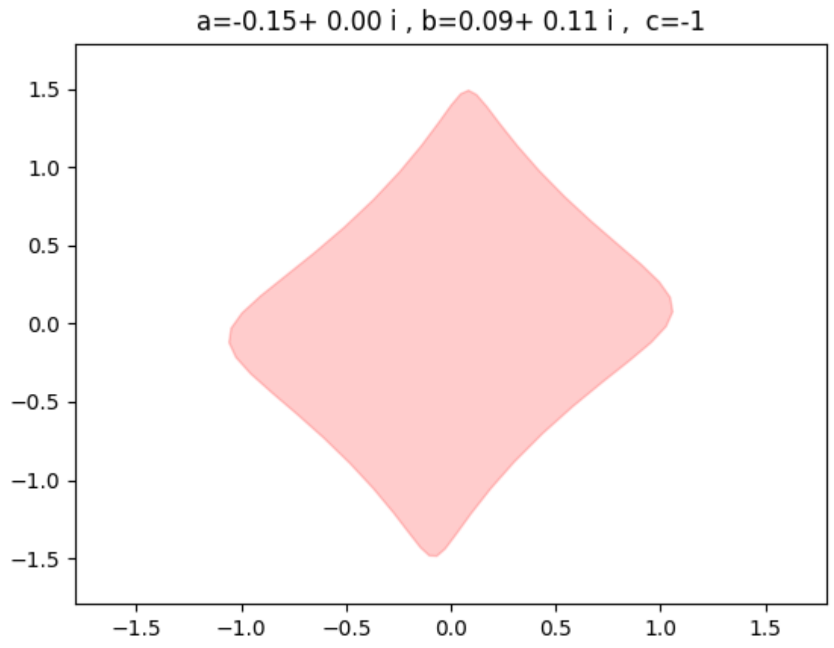
Example quartic

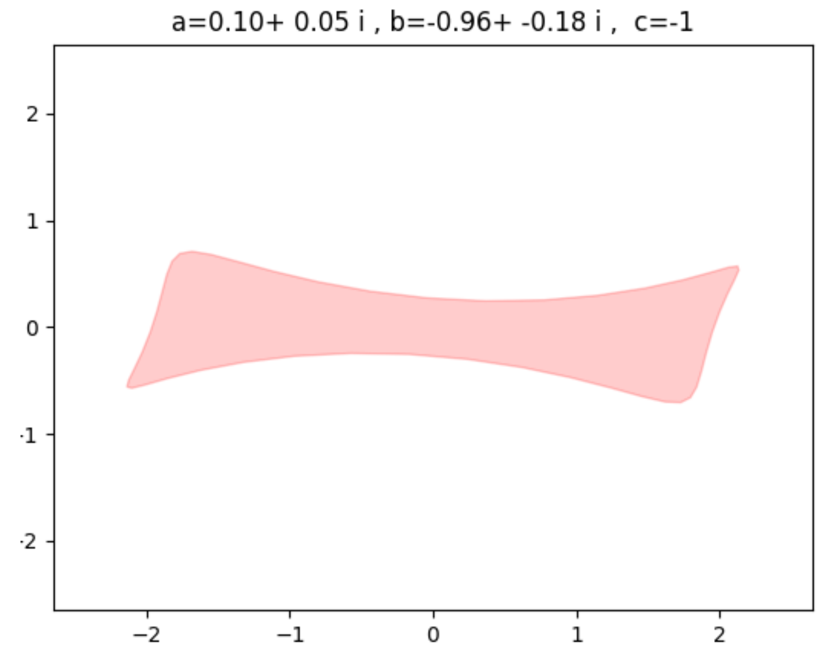
$$V(x) = a\frac{x^4}{4} + b\frac{x^2}{2} \qquad \bar{V}(y) = \bar{a}\frac{y^4}{4} + \bar{b}\frac{y^2}{2} \qquad \prod_i e^{-N(-c|\lambda_i|^2 + 2\Re V(\lambda_i))} \prod_{i<j} |\lambda_i - \lambda_j|^2$$

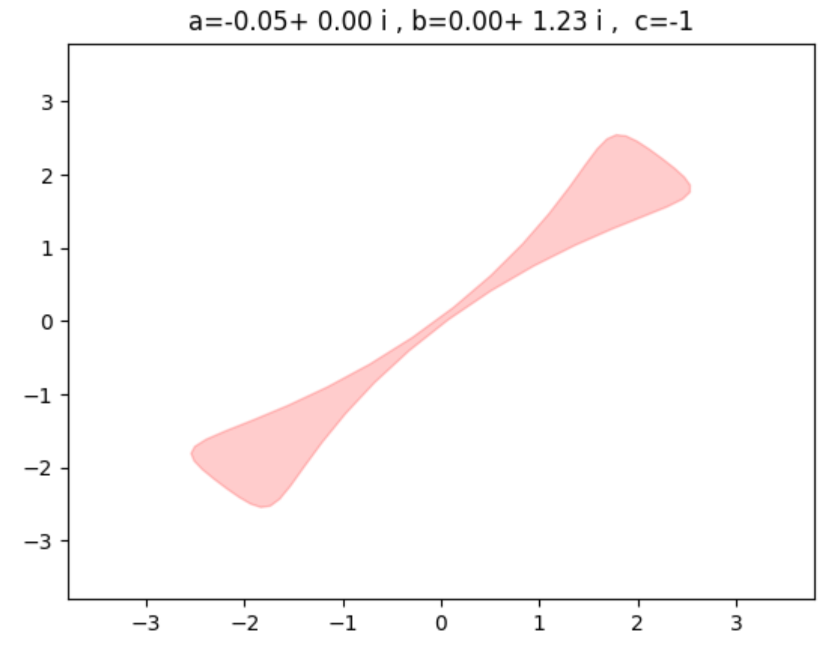
$$E(x,y) = (ax^3 + bx - cy)(\bar{a}y^3 + \bar{b}y - cx) - \frac{1}{c}(a\bar{a}x^2y^2 + px^2 + \bar{p}y^2 + K) + c$$

Find $p \in \mathbb{C}, K \in \mathbb{R}$ that minimise energy

Domain D of boundary ∂D $E(z, \bar{z}) = 0$







Asymptotic expansion

Asymptotic expansion

Asymptotic expansion

Asymptotic expansion

Large *N*

Limit density(ies) that minimise energy

Asymptotic expansion

Large *N*

Limit density(ies) that minimise energy

- 1-matrix : $\rho(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2} = \frac{1}{2\pi i}\text{Discontinuity}(\text{Y}(\text{x}))$

Resolvent: $W(x) = V'(x) - Y(x)$ $E(x, Y(x)) = 0 = Y^2 - V'(x)Y + P(x)$

Asymptotic expansion

Large N

Limit density(ies) that minimise energy

- 1-matrix : $\rho(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2} = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$

  Resolvent: $W(x) = V'(x) - Y(x)$ $E(x, Y(x)) = 0 = Y^2 - V'(x)Y + P(x)$

- 2-matrix : $\rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$ $\tilde{\rho}(y) = \frac{1}{2\pi i}\text{Discontinuity(X(y))}$

  $E(x, Y(x)) = 0$ $E(X(y), y) = 0$

  $E(x, y) = (V'(x) - cy)(\tilde{V}'(y) - cx) - \frac{1}{c}P(x, y) + c$

  Resolvent: $W(x) = V'(x) - cY(x)$ $\tilde{W}(y) = \tilde{V}'(y) - cX(y)$

Asymptotic expansion

Large N

Limit density(ies) that minimise energy

- 1-matrix : $\rho(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2} = \frac{1}{2\pi i}\mathrm{Discontinuity(Y(x))}$

  Resolvent: $W(x) = V'(x) - Y(x)$ $E(x, Y(x)) = 0 = Y^2 - V'(x)Y + P(x)$

- 2-matrix : $\rho(x) = \frac{1}{2\pi i}\mathrm{Discontinuity(Y(x))}$ $\tilde{\rho}(y) = \frac{1}{2\pi i}\mathrm{Discontinuity(X(y))}$

  $E(x, Y(x)) = 0$ $E(X(y), y) = 0$

  $E(x, y) = (V'(x) - cy)(\tilde{V}'(y) - cx) - \frac{1}{c}P(x, y) + c$

  Resolvent: $W(x) = V'(x) - cY(x)$ $\tilde{W}(y) = \tilde{V}'(y) - cX(y)$

- complex-matrix : $\rho(x) = \frac{-1}{c\pi}\chi_{\mathscr{D}}$ $\partial\mathscr{D}$ of equation $E(z, \bar{z}) = 0$

  Schwartz function $\bar{z} = Y(z)$

  Resolvent: $W(z) = V'(z) - c\bar{z}$

Asymptotic expansion

Large N

Limit density(ies) that minimise energy

- 1-matrix : $\rho(x) = \frac{1}{2\pi}\sqrt{4P(x) - V'(x)^2} = \frac{1}{2\pi i}\mathrm{Discontinuity(Y(x))}$

  Resolvent: $W(x) = V'(x) - Y(x)$ $\quad E(x, Y(x)) = 0 = Y^2 - V'(x)Y + P(x)$

- 2-matrix : $\rho(x) = \frac{1}{2\pi i}\mathrm{Discontinuity(Y(x))}$ $\quad \tilde{\rho}(y) = \frac{1}{2\pi i}\mathrm{Discontinuity(X(y))}$

  $E(x, Y(x)) = 0 \qquad E(X(y), y) = 0$

  $$E(x,y) = (V'(x) - cy)(\tilde{V}'(y) - cx) - \frac{1}{c}P(x,y) + c$$

  Resolvent: $W(x) = V'(x) - cY(x)$ $\quad \tilde{W}(y) = \tilde{V}'(y) - cX(y)$

- complex-matrix : $\rho(x) = \frac{-1}{c\pi}\chi_{\mathscr{D}}$ $\quad \partial\mathscr{D}$ of equation $E(z, \bar{z}) = 0$

  Schwartz function $\bar{z} = Y(z)$

  Resolvent: $W(z) = V'(z) - c\bar{z}$

Asymptotic expansion

Asymptotic expansion

Large *N*

Asymptotic expansion

Large N

- In all cases

Asymptotic expansion

Large *N*

- In all cases

$$W(x) = V'(x) - Y(x) = W_{0,1}(x)$$

Asymptotic expansion

Large N

- In all cases $W(x) = V'(x) - Y(x) = W_{0,1}(x)$ $E(x, Y(x)) = 0$

Asymptotic expansion

Large N

- In all cases

$$W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$$

Resolvent: $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right)$

Asymptotic expansion

Large *N*

- In all cases $$W(x) = V'(x) - Y(x) = W_{0,1}(x)$$ $$E(x, Y(x)) = 0$$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\operatorname{Tr}\frac{1}{x-\Lambda}\right) \quad \sim W(x)$$

Asymptotic expansion

Large *N*

- In all cases $$W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2} W_{1,1}(x)$$

Asymptotic expansion

Large *N*

- In all cases $$W(x) = V'(x) - Y(x) = W_{0,1}(x)$$ $$E(x, Y(x)) = 0$$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) \quad + \frac{1}{N^2} W_{1,1}(x) \quad + \frac{1}{N^4} W_{2,1}(x)$$

Asymptotic expansion

Large N

- In all cases $W(x) = V'(x) - Y(x) = W_{0,1}(x)$ $\quad E(x, Y(x)) = 0$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$$

Asymptotic expansion

Large *N*

- In all cases

$$W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$$

Assume series expansion *(often exists)*

Asymptotic expansion

Large N

- In all cases

$$W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations

Asymptotic expansion

Large N

- In all cases $\quad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$

Resolvent: $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations $\Longrightarrow$ recursion relation for

Asymptotic expansion

Large N

- In all cases $\quad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$

Resolvent: $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations $\Longrightarrow$ recursion relation for $W_{g,n}$

Asymptotic expansion

Large *N*

- In all cases $W(x) = V'(x) - Y(x) = W_{0,1}(x)$ $E(x, Y(x)) = 0$

Resolvent: $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations $\Longrightarrow$ recursion relation for $W_{g,n}$ = ***Topological Recursion***

Asymptotic expansion

Large N

- In all cases

$$W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$$

Resolvent: $$\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations $\Longrightarrow$ recursion relation for $W_{g,n}$ = ***Topological Recursion***

Topological Recursion = algebro-geometric recursion formula

Asymptotic expansion

Large N

- In all cases $\qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$

Resolvent: $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \frac{1}{N^4}W_{2,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations $\Longrightarrow$ recursion relation for $W_{g,n}$ = ***Topological Recursion***

Topological Recursion = algebro-geometric recursion formula

Knowing $W_{0,1}$ determines all the $W_{g,n}$

Asymptotic expansion

Large N

- In all cases $\qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad E(x, Y(x)) = 0$

Resolvent: $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) \quad + \frac{1}{N^2}W_{1,1}(x) \quad + \frac{1}{N^4}W_{2,1}(x) \quad + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

Assume series expansion *(often exists)*

Then:

Schwinger Dyson equations $\Longrightarrow$ recursion relation for $W_{g,n}$ = ***Topological Recursion***

Topological Recursion = algebro-geometric recursion formula

Knowing $W_{0,1}$ determines all the $W_{g,n}$

Conclusion

Conclusion

Conclusion

Conclusion

- Limit shape or density

Conclusion

- Limit shape or density minimize energy

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange
  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange
  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity}(\text{Y}(\text{x}))$$

- Complex case = special case of 2-matrix model

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\mathrm{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

  $$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange
  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model
  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$
- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right)$

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

  $E(x, Y(x)) = 0$ $W(x) = V'(x) - Y(x) = W_{0,1}(x)$ $\rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\text{Tr}\,\frac{1}{x - \Lambda}\right) \sim W(x)$

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange
    algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model
    Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x)$

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

  $$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity}(\mathrm{Y(x)})$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties   KdV, integrability

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties   KdV, integrability   Virasoro, W-algebra

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties KdV, integrability Virasoro, W-algebra Modular forms

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\text{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties KdV, integrability Virasoro, W-algebra Modular forms …etc

Conclusion

- Limit shape or density   minimize energy   unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties   KdV, integrability   Virasoro, W-algebra   Modular forms   …etc

Thank you for your attention

Conclusion

- Limit shape or density minimize energy unique solution of Euler Lagrange

  algebraic

$$E(x, Y(x)) = 0 \qquad W(x) = V'(x) - Y(x) = W_{0,1}(x) \qquad \rho(x) = \frac{1}{2\pi i}\text{Discontinuity(Y(x))}$$

- Complex case = special case of 2-matrix model

  Constant density in a domain bounded by the curve $E(z, \bar{z}) = 0$

- Asymptotic expansion $\frac{1}{N}\mathbb{E}\left(\mathrm{Tr}\,\frac{1}{x-\Lambda}\right) \sim W(x) + \frac{1}{N^2}W_{1,1}(x) + \ldots + \sum_g \frac{1}{N^{2g}}W_{g,1}(x)$

  Topological Recursion

  Knowing $W_{0,1}$ determines all the $W_{g,n}$

- Other properties KdV, integrability Virasoro, W-algebra Modular forms …etc

Thank you for your attention