"Escaping the crowd": emergent outliers in rank-1 non-normal deformations of GUE/CUE [1]

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[1] Based on: joint work with **Boris Khoruzhenko** and **Mihail Poplavsky**, arXiv:2211.00180

Consider two ensembles described by $N \times N$ random matrices J which are **non-selfadjoint** and **non-normal**. The first one is of the form

$$J_{GUE} = H + i\gamma \,\text{diag}(1, 0, \ldots, 0) \equiv H + i\gamma\, \mathbf{e} \otimes \mathbf{e}^T, \quad \gamma \geq 0, \text{ and } \mathbf{e}^T = (1, 0, \ldots, 0)$$

where H is **complex Hermitian** $\in GUE$, with N real eigenvalues X_i whose density converges as $N \to \infty$ to the Wigner semicircle: $\nu(X) = \frac{1}{2\pi}\sqrt{4 - X^2}$.
All eigenvalues z_i of J_{GUE} are distributed for $\gamma > 0$ in the **upper half** of the complex plane $\Im z > 0$, with the joint probability density (**YF** & **B. Khoruzhenko** '99):

$$\mathcal{P}_N \{z_i\} = \frac{1}{h_{\beta,N}} e^{-\frac{\beta N}{4}\sum_{i=1}^{N} Re\,(z_i^2)} \times \prod_{1 \leq j < k \leq N} |z_j - z_k|^2 \,\delta(\sum_{i=1}^{N} Im\, z_j - \gamma)$$

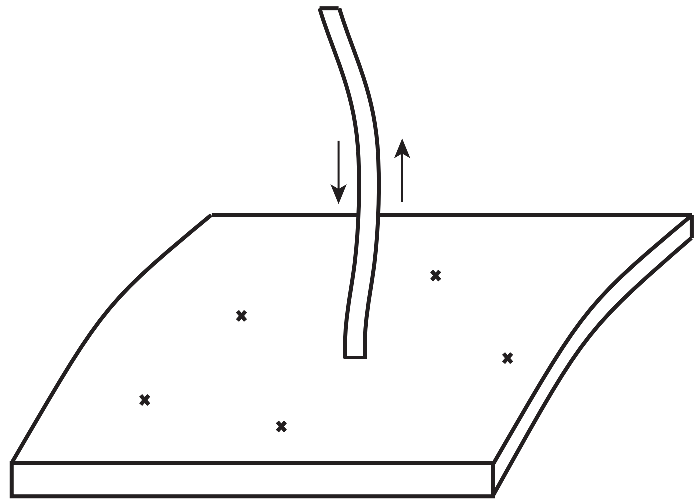
Similarly, the second ensemble is defined as

$$J_{CUE} = \hat{U}\,\text{diag}(\sqrt{1-T}, 1, \ldots, 1), \quad 0 \leq T \leq 1\,,$$

with $N \times N$ matrix $\hat{U} \in CUE$ is Haar-distributed **complex unitary** with unimodular eigenvalues $e^{i\theta_i}$, θ_i are spaced randomly in $[0, 2\pi]$, with mean spacing $\Delta = 2\pi/N$.
All eigenvalues z_i of J_{CUE} for $T > 0$ belong to the **interior of the unit circle** $|z| < 1$ with the JPD (**YF**' 00)

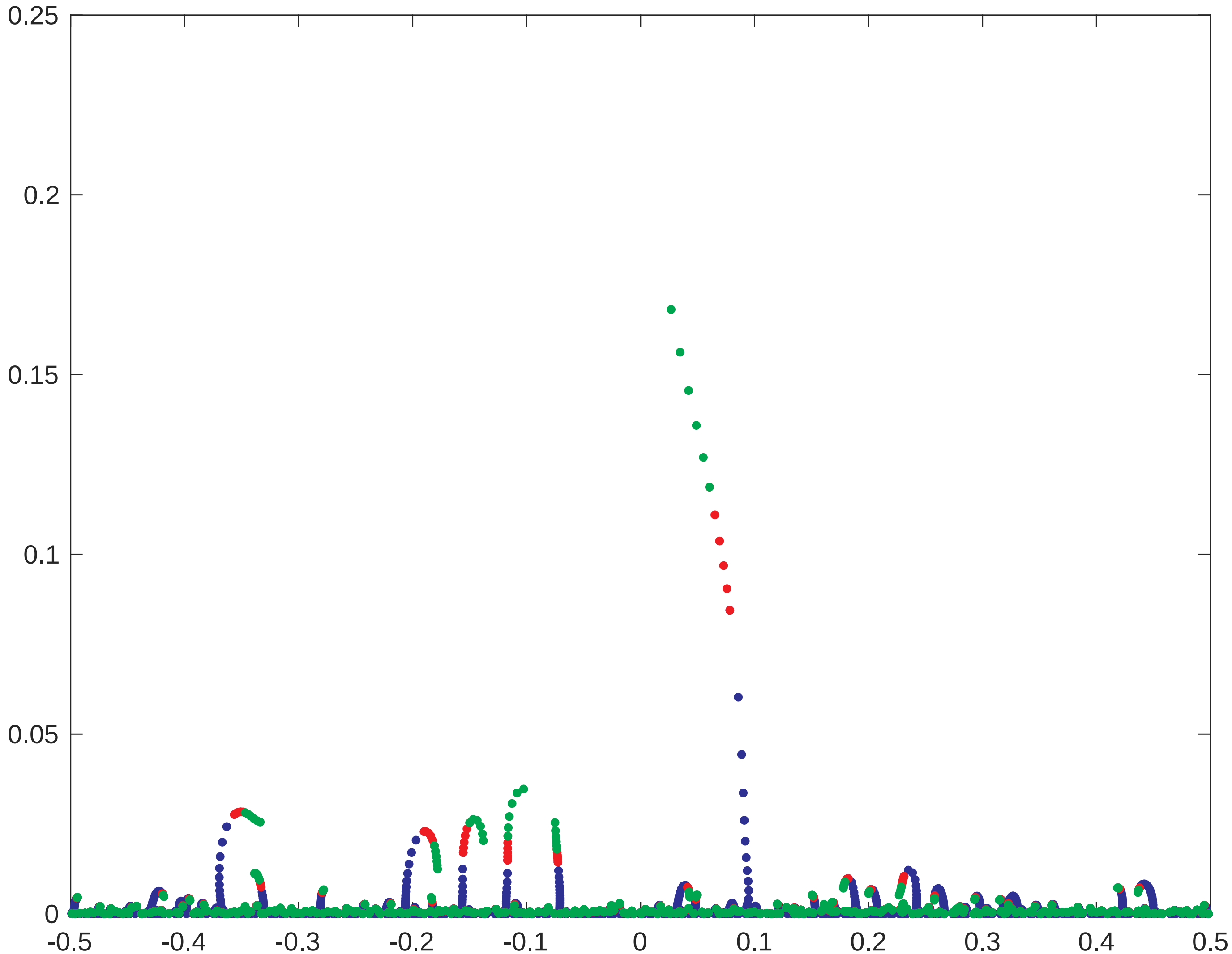
$$\mathcal{P}_N \{z_i\} = \frac{1}{N\pi^N T^{N-1}} \prod_{1 \leq i < j \leq N} |z_j - z_i|^2 \,\delta\Big(1 - T - \prod_{j=1}^{N} |z_j|^2\Big).$$

Remark: Both ensembles emerge in a random matrix description of wave scattering from a **chaotic domain**, cf. **Verbaarschot**, **Weidenmueller**, **Zirnbauer** '85



Escape to infinity from the "inner" domain can be effectively described by a **non-Hermitian Hamiltonian** $J_{eff} = H - i\gamma \mathbf{e} \otimes \mathbf{e}^T, \ \gamma \geq 0$.
Similar meaning can be attributed to **subunitary** J_{CUE} describing time-periodic evolution under open **quantum chaotic maps**.

In the physics literature the complex eigenvalues z_i of J_{eff} are associated with **poles** of the **scattering matrix**, known as "**resonances**", with $\Im z_i$ called the "**resonance width**" . In that context it has been predicted heuristically by **F.-M.Dittes** et al. '91 that at $\gamma = 1$ an **abrupt restructuring** of resonance widths in the complex plane should take place, with a single **outlier** of **finite width** $\gamma - \gamma^{-1}$ emerging for $\gamma > 1$. Rigorously confirmed: **O'Rourke** $\&$ **Wood**' 17; **J. Rochet** '17.



Trajectories of eigenvalues of the matrix $H + i\gamma \mathbf{e} \otimes \mathbf{e}^T$ of dimension $N = 1000$ as functions of γ. The **blue** dots correspond to values in $0 \leq \gamma \leq 0.5$ the **red** dots to values in $0.5 \leq \gamma \leq 1$ and the **green** dots to values $1.0 \leq \gamma \leq 1.5$ with increments 0.01.

Our main goal is to get a closer insight into **extreme values** and **emergence** of the **outlier**, and develop the quantitative understanding of the **restructuring transition** and **trapping phenomenon**.

A reminder on Extreme Value Statistics for i.i.d. unbounded positive variables:

Consider a set of N real random i.i.d. variables $x_1, \ldots, x_N$ with **parent probability density** $p(x)$ supported on the whole positive semi-axis $x \in [0, \infty)$. Define

$$M = max\{x_1, \ldots, x_N\} \quad \text{and} \quad Q_N(x) = Prob\,[M \leq x]$$

Question: Does $Q_N(x)$ show any **universality** as $N \to \infty$?

It turns out only **two** possibilities arise:

- If the parent density decays as a powerlaw: $p(x \gg 1) \sim Ax^{-(1+\alpha)}$ with $A, \alpha > 0$, then after a rescaling Q_N converges to the **Fréchet** limiting form:

$$\lim_{N\to\infty} Q_N\left(b_N z\right) = e^{-z^{-\alpha}}\theta(z) \text{ with } b_N = (AN/\alpha)^{\frac{1}{\alpha}}$$

- If the parent density decays faster as any power: $p(x \gg 1) \sim e^{-x^{\delta}}$ with $\delta > 0$, then after a rescaling and a shift Q_N converges to the **Gumbel** limiting form:

$$\lim_{N\to\infty} Q_N\left(a_N + b_N z\right) = e^{-e^{-z}} \text{ with } a_N = (\ln N)^{\frac{1}{\delta}} \text{ and } b_N = \delta^{-1}\,(\ln N)^{\frac{1}{\delta}-1}$$

To get insights into **e.v.s.** of resonances, we concentrate on the **mean density of imaginary parts** for complex eigenvalues $z_i = X_i + iY_i$, defined as

$$\rho_N^{(im)}(Y) = \left\langle \frac{1}{N}\sum_{i=1}^{N} \delta(Y - \Im z_i) \right\rangle$$

Depending on the value of Y this function has a very different behaviour.

(**I**) Let Y to be of the same order as the **mean eigenvalue spacing** $\Delta \sim N^{-1}$ in the *horizontal* direction, that is $Y = y/N$ while keeping y fixed as $N \to \infty$. Then

$$\lim_{N\to\infty} \frac{1}{N}\rho_N^{(im)}\left(Y = \frac{y}{N}\right) = -\frac{d}{dy}\left[\frac{e^{-y\left(\gamma+\frac{1}{\gamma}\right)}}{y} I_1(2y)\right] := \rho_\infty(y),$$

where $I_p(z)$ is the modified Bessel function. One can infer that

(**a**) $\int_0^\infty \rho_\infty(y) y\, dy = \gamma$ for $\gamma < 1$ in full agreement with the exact sum rule $\sum \Im z_i = \gamma$. However, for $\gamma > 1$ one finds $\int_0^\infty \rho_\infty(y) y\, dy = \frac{1}{\gamma} < \gamma$. The **sum rule deficit** $\gamma - \frac{1}{\gamma}$ suggests that for $\gamma > 1$ some eigenvalues are "**missing**" at the scale $1/N$.

(**b**) $$\rho(y \gg 1) \sim \begin{cases} y^{-3/2} e^{-y\left(\gamma+\frac{1}{\gamma}-2\right)} & \text{for } \gamma \neq 1 \\ y^{-5/2} & \text{for } \gamma = 1 \end{cases}$$

hinting to a **Gumbel**-type e.v.s for $\gamma \neq 1$ but **Fréchet**-type for $\gamma = 1$ with $Y_{max} \sim N^{-1/3}$.

To search for missing eigenvalues one should look at scales of imaginary parts such that $N^{-1} \ll Y < \gamma$ as $N \to \infty$. One can show that:

(**II**) for every fixed γ the density $\rho_N^{(im)}(Y)$ has a **Large Deviation** form:

$$\rho_N^{(im)}(Y) = \Psi_\gamma(Y) e^{-N\Phi_\gamma(Y)}, \quad Y \in [0, \gamma)$$

where the **rate function** is given by

$$\Phi_\gamma(Y) = Y(\gamma - Y) - \ln \frac{\gamma - Y}{\gamma} - Y r_*(Y) + 2 \ln r_*(Y),$$

with $r_*(Y) = \frac{\sqrt{Y^2+4} - Y}{2}$ and $\Psi_\gamma(Y)$ is also explicitly known.

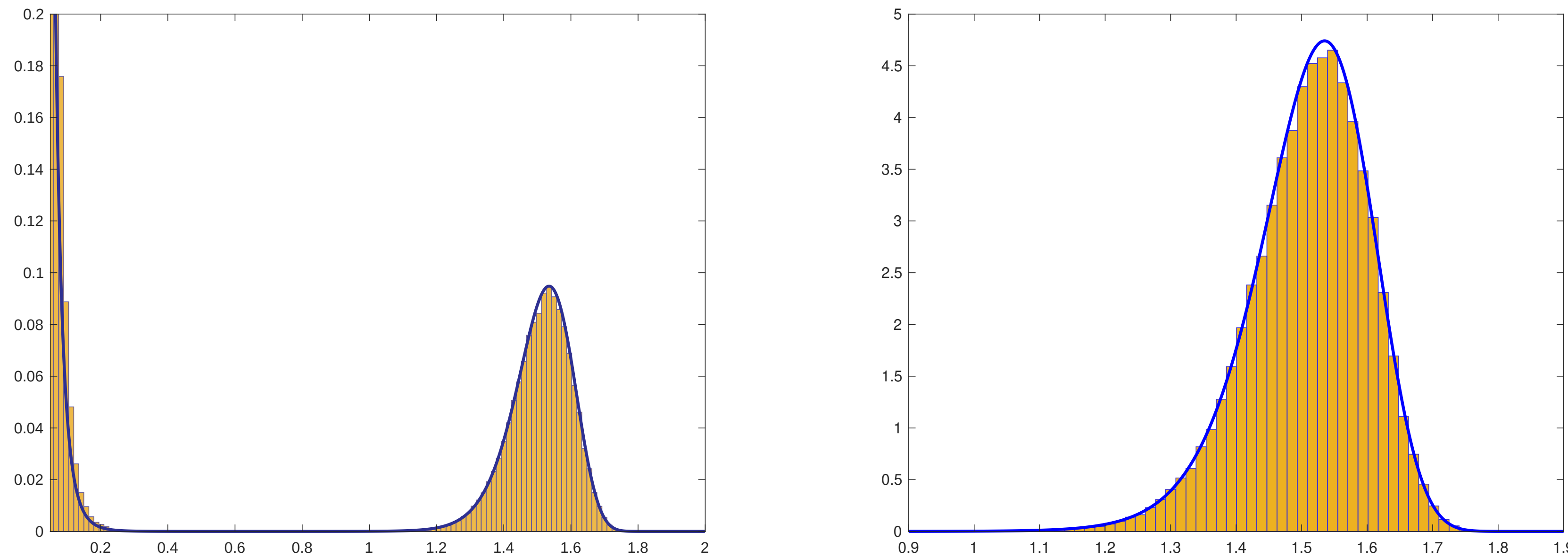
It further turns out that:

(**a**) for $\gamma < 1$ the function $\frac{d}{dY}\Phi_\gamma(Y) > 0, \ \forall Y \in [0, \gamma)$.

(**b**) $\Phi_\gamma(Y)$ becomes non-monotonic for $\gamma > 1$ and has the **global minimum** $\Phi_\gamma(Y_*) = 0$ at $Y_* = \gamma - \frac{1}{\gamma}$ and the **local maximum** at $Y_{**} = \frac{2\left(\gamma - \frac{1}{\gamma}\right)}{3 + \sqrt{1 + \frac{8}{\gamma^2}}} < Y_*$.

Moreover, the pre-exponential factor $\Psi_\gamma(Y)$ vanishes at Y_{**}.

Remark: For $\gamma > 1$ the value $\Im z_i = Y_*$ is the most probable value for the " resonance width" in the region of "**wide resonances**” $Y \gg 1/N$, defining for $N \gg 1$ a single **outlier**. At the same time, the value $Y = Y_{**}$ can be interpreted as the **true boundary** between the **outlier** and the ”**sea of narrow resonances**” extending from the scale $Y \sim 1/N$ to $Y = Y_{**}$.



Histogram of the distribution of the
(**a**) imaginary parts of the eigenvalues z_j of $J = H + 2i\,\text{diag}(1, 0, \ldots, 0)$ versus the Large Deviation approximation $\rho_N^{(im)}(Y)$. Note: $Y_{**} \approx 0.634$ and $Y_* = 1.5$.
(**b**) the largest imaginary part $Y_{max} = max_{j=1\ldots N} \Im z_j$ versus $N\rho_N^{(im)}(Y)$.
Each plot was produced from 100,000 realisations of GUE matrix of dimension $N = 50$. One can observe positive skewness in the fluctuations of Y_{max} for finite matrix dimensions beyond the Gaussian approximation for LDP in the vicinity of $Y = Y_* = \gamma - \frac{1}{\gamma}$ given by:

$$\rho_N^{(im)}(Y) \approx \frac{1}{N\sqrt{2\pi\sigma^2}} e^{-\frac{(Y-Y_*)^2}{2\sigma^2}}, \quad \sigma^2 = \frac{1}{N\gamma^2}\frac{\gamma^2+1}{\gamma^2-1}\,.$$

The value $\gamma = 1$ is **critical** as the **outlier** merges with the "**sea**".

As was shown by **G. Dubach** $\&$ **L. Erdös**'21 the outlier is still **distinguishable** from the sea for $\gamma - 1 > N^{-1/3+\epsilon}$, $\forall \epsilon$ suggesting the **critical scaling** $N^{-1/3}$.

We develop a more detailed picture.

Theorem (**YF**, **Khoruzhenko** $\&$ **Poplavskyi** '22). *Consider the scaling regime* $\gamma = 1 + \frac{\alpha}{N^{1/3}}$*, where the parameter* $\alpha \in \mathbb{R}$ *is fixed. Then for* $Y = \frac{m}{N^{1/3}}$ *with fixed* $m > 0$ *the following limit exists:*

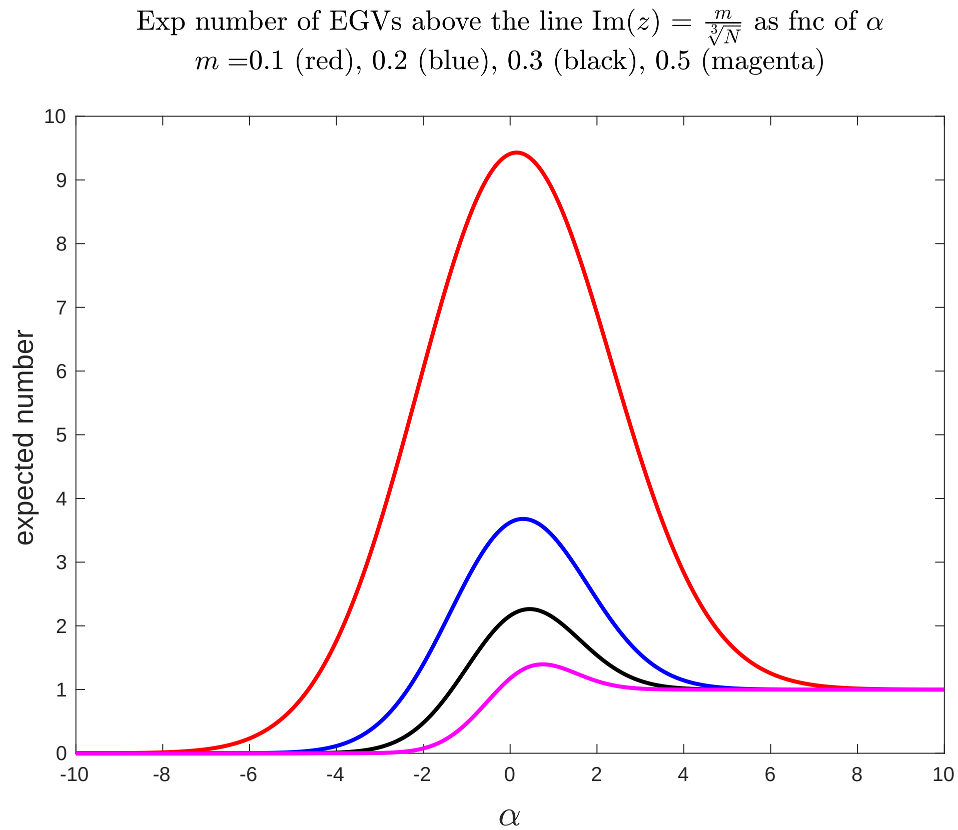
$$\lim_{N\to\infty} \left[N \rho_N^{(im)}(Y) \frac{dY}{dm} \right] = \frac{1}{2\sqrt{\pi}} \frac{\left[\frac{3}{2m} + \left(\frac{3m}{2} - \alpha\right)^2\right]}{m^{3/2}} e^{-m\left(\alpha - \frac{m}{2}\right)^2} := \tilde{\rho}(m), \quad m > 0\,.$$

Remark 1: At $N \to \infty$ the mean number of eigenvalues with imaginary parts $\Im z$ exceeding $Y = \frac{m}{N^{1/3}}$ is given by $\int_m^\infty \tilde{\rho}(m)\, dm = O(1)$.

This implies that $N^{-1/3} \gg \Delta = 1/N$ is indeed the correct scale of the **extreme values** for imaginary parts in the critical regime.

Remark 2: We can separately show that only eigenvalues with **real parts** in a narrow window of the widths $|\Re z_i| \sim N^{-1/3} \ll 1$ around the origin contribute to **extreme values** at the scale $\Im z \sim N^{-1/3}$.

Trapping phenomenon: the expected number of eigenvalues with imaginary parts $\Im z$ exceeding the level $\frac{m}{N^{1/3}}$ as function of α developes a maximum:

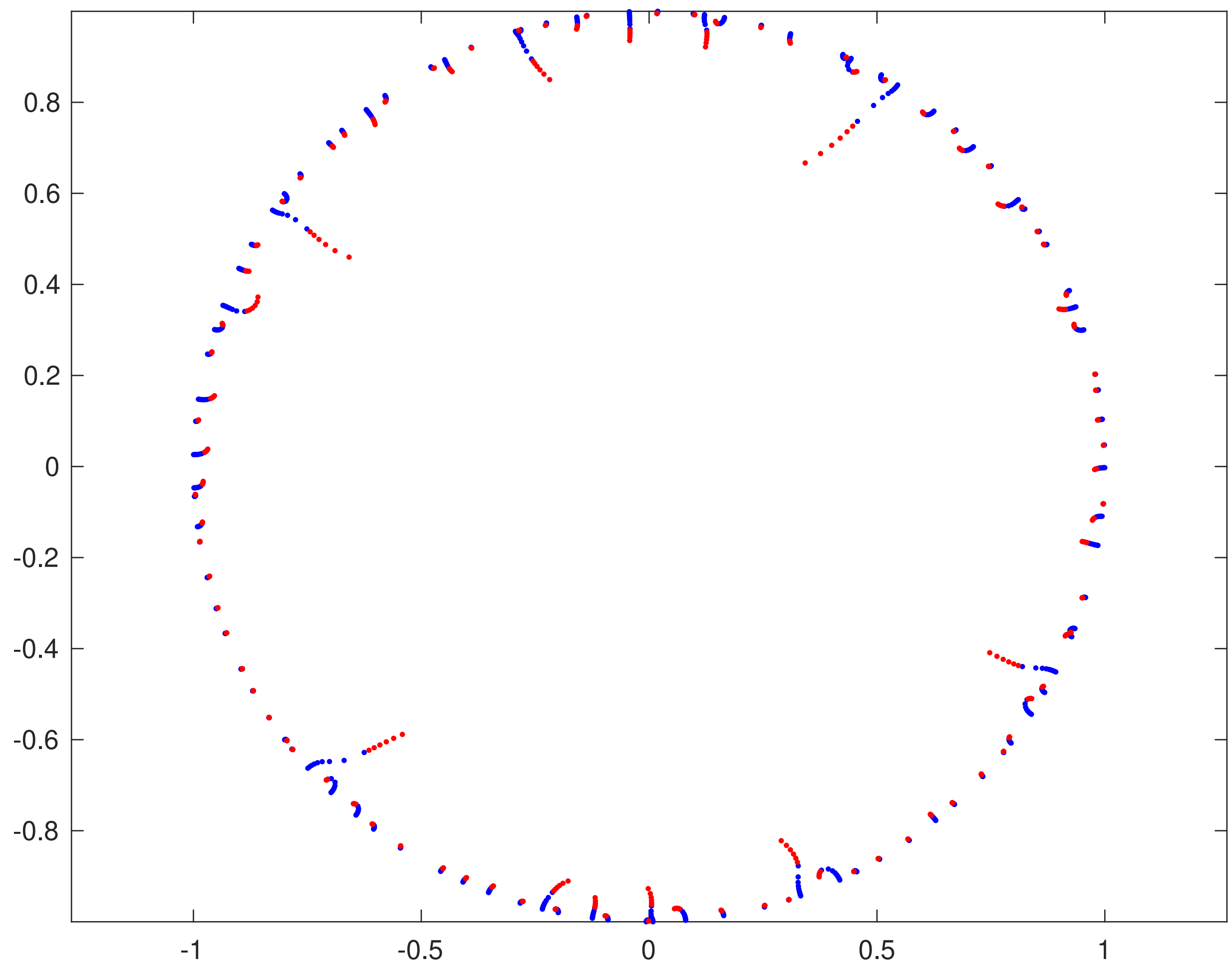


Challenge remaining: finding the distribution of the largest imaginary part in the scaling regime.

This can be achieved for the model of **subunitary** matrices:

$$J_{CUE} = \hat{U}\,\mathrm{diag}(\sqrt{1-T}, 1, \ldots, 1), \quad 0 < T \leq 1.$$

Trajectories of eigenvalues of the matrix $J_{CUE} = \hat{U}$ diag$(\sqrt{1-T}, 1, \ldots, 1)$, of dimension $N = 100$ as functions of $\tau = \sqrt{1-T}$.



The **blue** dots correspond to values of τ in $1 \geq \tau \geq 0.2$, the **red** dots to values in $0.2 \geq \tau \geq 0.08$ with decrements 0.01. One can see **emergence** of a few **outliers** when $\tau \sim N^{-1/2}$ (equivalently $1-T \sim N^{-1}$). Our main goal is to get a closer insight into the associated **Extreme Value Statistics** (EVS) in the critical regime $T = 1 - \frac{t}{N}$, with t fixed as $N \to \infty$.

The quantity of our primary interest is the eigenvalue z_i **closest to the origin**, i.e. with the smallest modulus

$$x_{min} := \min_{j=1,\ldots,N} |z_j|$$

Note that for $T = 1$ the matrix J_{CUE} has exactly zero eigenvalue, hence $x_{min} = 0$
Our main result is the following

Theorem (**YF** & **Khoruzhenko** '22+). *In the scaling limit $N \to \infty$ keeping $t = N(1-T) > 0$ fixed, the smallest modulus x_{min} converges weakly to a random variable X whose cumulative distribution function $F_X^{(t)}(x) =$ Prob$\{X \le x\}$ is given by the series*

$$F_X^{(t)}(x) = e^t \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n(n-1)} \exp\{-\frac{t}{x^{2n}}\}}{\prod_{k=1}^{n}(1-x^{2k})} \qquad (0 < x < 1).$$

Remark. This is different from the standard laws due to **Gumbel**, **Fréchet** and **Weibull** characterising the extreme values in long sequences of i.i.d. random variables.

We can further show that as the parameter t changes over $\mathbb{R}_+$, the distribution $F_X^{(t)}(x)$ interpolates between **Fréchet** and **Gumbel** distributions. Namely:

As $t \to 0$ typically $x_{min} \sim \sqrt{t}$ and $\lim_{t\to 0} \Pr(x_{min}/\sqrt{t} < y) = \exp\{-y^{-2}\}$.

whereas in the limit $t \gg 1$ x_{min} is Gumbel-distributed:

$$x_{min} = 1 - \frac{\ln t - \ln(\ln t) + \mathbf{Gumbel}}{2t}$$

Summary:

For the model $J_{GUE} = H + i\gamma \,\text{diag}\,(1,0,\ldots,0)$ of non-selfadjoint matrices we provided a detailed description of the mean eigenvalue density **restructuring** (aka "**trapping transition**") in the region of **extreme imaginary parts** in the complex plane as the function of coupling γ, happening in the critical region $\gamma - 1 \sim N^{-1/3}$. In a related model of subunitary $J_{CUE} = \hat{U}\,\text{diag}(\sqrt{1-T}, 1, \ldots, 1), \quad 0 \le T \le 1$ we were able to compute explicitly the distribution of the eigenvalue with the smallest modulus, finding that in the critical regime $1 - T \sim N^{-1}$ it is described by distribution nontrivially interpolating between **Gumbel** and **Fréchet**.

Remark: there is a clear analogy between the restructuring of resonances and the **condensation transition** in models of mass transport, when the globally conserved mass M exceeds a critical value, see e.g. **Majumdar** arXiv.0904.4097.

Open problems: Extremes and outliers - statistics and universality?

Extension of JPD to general β is known (**Kozhan'**17)

$$\mathcal{P}_z\{z_i\} = \frac{1}{h_{\beta,N}}\, e^{-\frac{\beta N}{4}\sum_{i=1}^{N} Re\,(z_i)^2}$$
$$\times \prod_{1\le j<k\le N} |z_j - z_k|^2 \prod_{j,k=1}^{N} |z_j - \overline{z}_k|^{\frac{\beta}{2}-1}\, \delta\Big(\sum_{i=1}^{N} Im\, z_j + \gamma\Big)$$

Perturbations of higher rank? Statistics of **left** $\&$ **right** eigenvectors? Experimental verifications?