Determinantal Coulomb gases in an uncharged region

(joint work with Raphael Butez, Alon Nishry and Aron Wennman)

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Coulomb gases and universality

Coulomb gas of n particles

Particles at $x_1, \ldots, x_n \in \mathbb{C}$ with charge $q_n > 0$ confined by V,

$$H_n(x_1, \ldots, x_n) = -{q_n}^2 \sum_{i<j}^{n} \log |x_i - x_j| + q_n \sum_{i=1}^{n} V(x_i).$$

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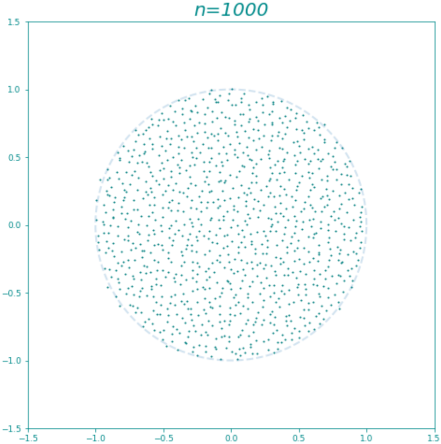
Determinantal setting $\beta_n {q_n}^2 = 2$,

$$\exp(-\beta_n H_n(x_1, \ldots, x_n)) = \prod_{i<j}^{n} |x_i - x_j|^2 \prod_{i=1}^{n} \exp\left(-\frac{2}{q_n} V(x_i)\right).$$

Ginibre ensemble example

$V(x) = \frac{|x|^2}{2}$ and $q_n = \frac{1}{n}$.



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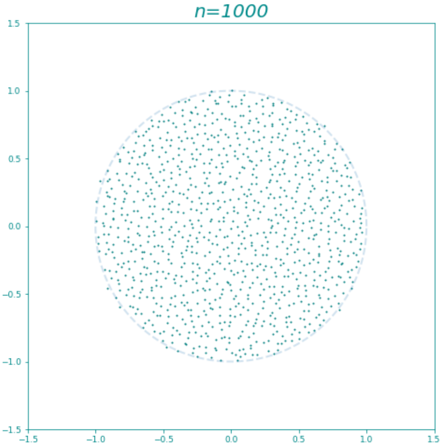

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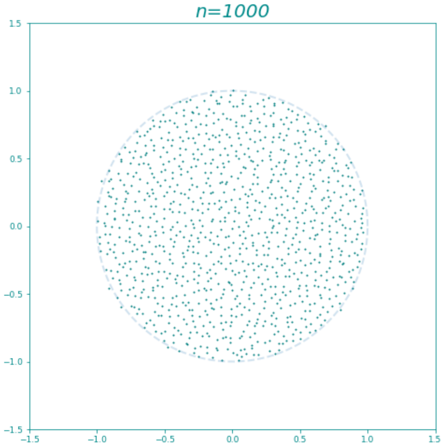

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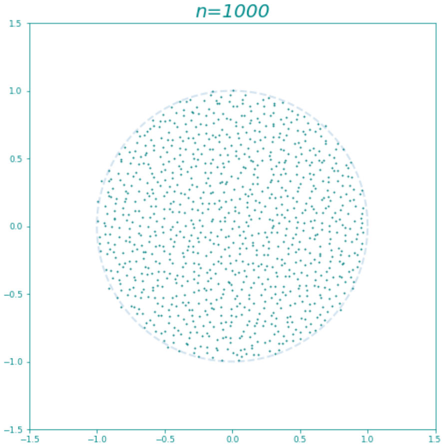

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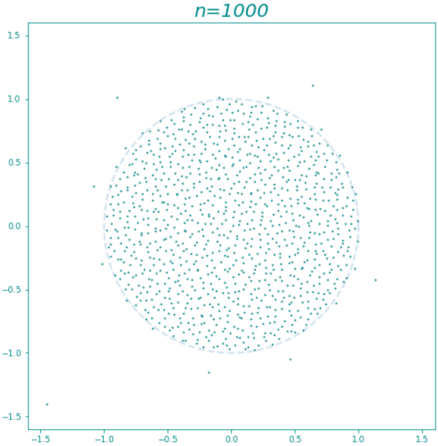
$$-\frac{\ell_{\mathbb{D}^c}}{\pi} \text{ survives} \implies \mathbb{D}^c \text{ empty!}$$

A weakly confining example

$$V(x) = \frac{|x|^2}{2} 1_{\mathbb{D}}(x) + (\log|x| + \tfrac{1}{2}) 1_{\mathbb{D}^c}(x) \text{ and } q_n = \frac{1}{n+1}.$$



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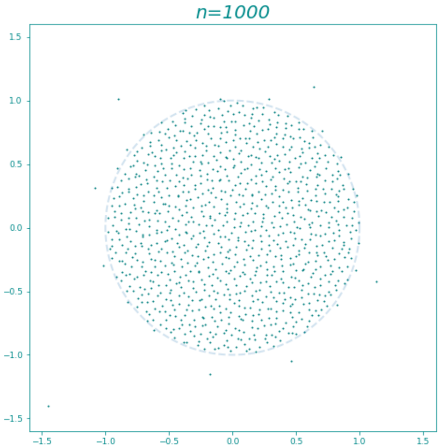

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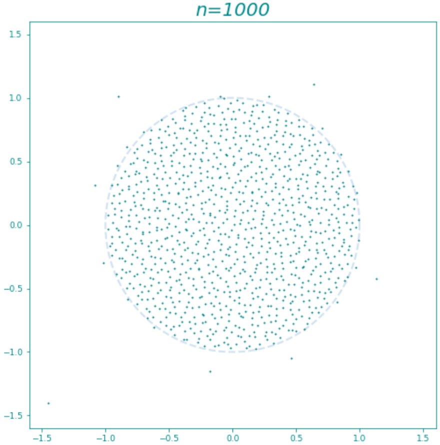

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$$q_n \sum_{k=1}^{n} \delta_{x_k} \longrightarrow \frac{\ell_{\mathbb{D}}}{\pi}.$$

No confining charge survives.

Confining background

Probability measure μ on $\mathbb{C}$. **Confining potential**:

$$V^{\mu}(x) = \int_{\mathbb{C}} \log|x-y| \mathrm{d}\mu(y) \quad \text{so that} \quad \Delta V^{\mu} = 2\pi\mu.$$

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Our Coulomb gas:

$$(X_1, \ldots, X_n) \sim \frac{1}{\mathcal{Z}_n} \prod_{i<j}^{n} |x_i - x_j|^2 \prod_{i=1}^{n} \exp\left(-\frac{2}{q_n} V^{\mu}(x_i)\right).$$

$$\text{Well-defined } (i.e., \mathcal{Z}_n < \infty) \quad \Longleftrightarrow \quad nq_n < 1.$$

If $nq_n \longrightarrow 1$ then

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Question:

If there is no charge in A, i.e., if $\mu(A) = 0$, what can be said about

the limit behavior of $\#A$?

Determinantal setting

Vandermonde determinant:

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$(z^{j-1})_{1\leq j\leq n} \rightsquigarrow (\varphi_j)_{1\leq j\leq n}$ basis of polynomials of degree $\leq n-1$.

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$$\prod_{i<j}^{n}(z_i - z_j) \propto \det\left((\varphi_j(z_i))_{1\leq i,j\leq n}\right).$$

$$\implies \quad \prod_{i<j}^{n}|x_i - x_j|^2 \prod_{i=1}^{n} e^{-\frac{2}{q_n}V^{\mu}(x_i)} \propto \det\left(K(x_i, x_j)_{i,j\leq n}\right)$$

where $K : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is given by

$$K(z,w) = \sum_{i=1}^{n} \varphi_i(z)\overline{\varphi_i(w)}e^{-\frac{1}{q_n}V^{\mu}(z)}e^{-\frac{1}{q_n}V^{\mu}(w)}.$$

Choose $(\varphi_j)_{1\leq j\leq n}$ such that

$$\int_{\mathbb{C}} \varphi_i(z)\overline{\varphi_j(z)}e^{-\frac{2}{qn}V^{\mu}(z)}\mathrm{d}\ell_{\mathbb{C}}(z) = \delta_{ij}.$$

Then, K is the kernel of the orthogonal projection onto $\mathbb{C}_{\leq n-1}[z]$.

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$$(X_1,\dots,X_n) \sim \frac{1}{n!}\det\left(K(x_i,x_j)_{i,j\leq n}\right)\mathrm{d}\ell_{\mathbb{C}}(x_1,\dots,x_n)$$

and, for $A_1,\dots,A_k \subset \mathbb{C}$ pairwise disjoint,

$$\mathbb{E}[\#A_1\cdots\#A_k] = \int_{A_1\times\cdots\times A_k} \det\left(K(x_i,x_j)_{i,j\leq k}\right)\mathrm{d}\ell_{\mathbb{C}^k}(x_1,\dots,x_k).$$

(correlation function) $\rho_k(x_1,\dots,x_k) = \det\left(K(x_i,x_j)_{i,j\leq k}\right).$

We shall suppose that $q_n \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$. Then,

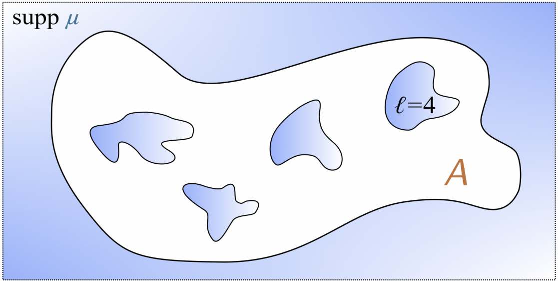
$$\mathbb{C}_{\leq n-1}[z] = L^2(\mathbb{C}, e^{-\frac{2}{q_n} V^\mu} \mathrm{d}\ell_{\mathbb{C}}) \cap \text{Holomorphic functions.}$$

So, K can be thought of as the kernel of the orthogonal projection

$$L^2(\mathbb{C}, e^{-\frac{2}{q_n} V^\mu} \mathrm{d}\ell_{\mathbb{C}}) \twoheadrightarrow L^2(\mathbb{C}, e^{-\frac{2}{q_n} V^\mu} \mathrm{d}\ell_{\mathbb{C}}) \cap \text{Holomorphic func.}$$

Outliers

Let A be a bounded connected component of $\mathbb{C} \setminus \operatorname{supp} \mu$.
Suppose that it has ℓ holes.



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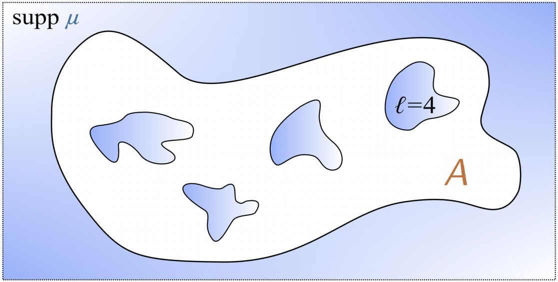

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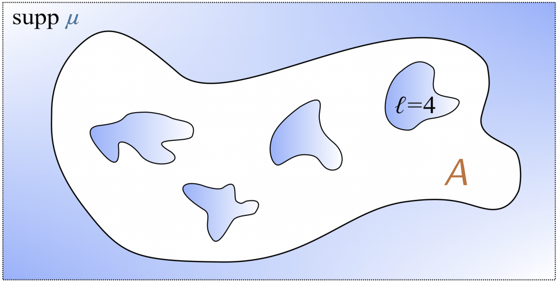

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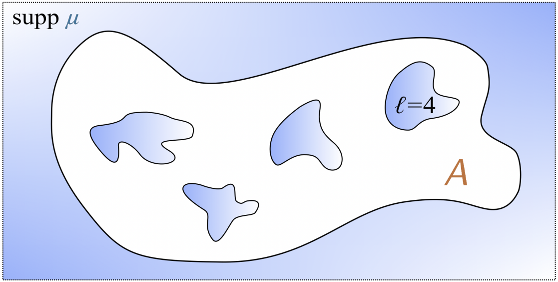

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If $\ell > 0$, the possible limits of $\rho_k^{(n)}|_{A^k}$ are indexed by $\mathbb{T}^\ell = (S^1)^\ell$.

Ideas

1. For $\Omega \subset \mathbb{C}$ open and $V : \Omega \to \mathbb{R}$, let $K : \Omega \times \Omega \to \mathbb{C}$,

$$K : L^2(\Omega, e^{-2V}\mathrm{d}\ell_\Omega) \twoheadrightarrow L^2(\Omega, e^{-2V}\mathrm{d}\ell_\Omega) \cap \text{Holomorphic func.}$$

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4. $\Delta V^\mu|_A = 0 \Rightarrow$ enough to classify the kernels for V harmonic.
5. Indexed by $\mathbb{T}^\ell = (S^1)^\ell$, up to the equivalence above.

Remarks

Fermions

Det. Coulomb gas $\simeq$ 2D fermion gas under $B = \Delta V$.
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Line bundles

Last step $\equiv$ classifying the flat Hermitian line bundles on A.

- e^{-2V} would be the metric,
- $\Delta V = 0$ the curvature and
- $|f|^2 e^{-2V}$ a transformation of the metric.

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Under some regularity conditions on μ, we can prove the following.

Theorem (Butez, G-Z, Nishry & Wennman, 2021)

For every $k \geq 1$,

$$\rho_k^{(n)}\big|_{A^k} \longrightarrow \rho_k.$$

Final remarks

- Coulomb gases in $\mathbb{R}$ where the interaction is $-|x-y|$. *Joint work with Chafaï and Jung (2022)*: Family indexed by S^1.

- Determinantal log-gases in $\mathbb{R}$. **Open question**.

- In $\mathbb{R}^d$ with $d \geq 3$ the exact analogue with μ a probability measure does not make sense (μ has to have infinite charge). In compact manifolds, the exact analogue does make sense. **Open question**.

- Raphael Butez, David García-Zelada, Alon Nishry, Aron Wennman.
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Thank you for your attention!

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