# Asymptotics of Matrix Models at low temperature

Alice GUIONNET

Based on a joint work with E. Maurel Segala

**Coulomb gases conference**

Post doc positions available, see http ://perso.ens-lyon.fr/aguionne/erc/

# Matrix models

It is the distribution of a $d$-tuple $\mathbf{X}^N = (X_1^N, \ldots, X_d^N)$ of $N \times N$ Hermitian matrices

$$d\mathbb{P}_N^V(\mathbf{X}^N) = \frac{1}{Z_N^V} e^{-N\mathrm{Tr}V(\mathbf{X}^N)} d\mathbf{X}^N$$

- $\mathrm{Tr}(A) = \sum A_{ii}$,
- $V$ is a self-adjoint polynomial :

$$V(X_1, \ldots, X_d) = \sum_{r=1}^{k} c_r X_{i_1^r} \cdots X_{i_{p_r}^r} = \sum_{r=1}^{k} \bar{c}_r X_{i_{p_r}^r} \cdots X_{i_{p_1}^r} ,$$

- $d\mathbf{X}^N = dX_1^N \cdots dX_d^N$ the Lebesgue measure on the entries

$$dX_i^N = \prod_{k \le \ell} d\Re(X_i(k\ell)) \prod_{k<\ell} d\Im(X_i(k\ell))$$

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Question : Does there exists $\tau_V$ s.t $\forall P \in \mathbb{C}\langle X_1, \ldots, X_d \rangle$

$$\int \frac{1}{N} \mathrm{Tr}\left(P(\mathbf{X}^N)\right) d\mathbb{P}_N^V(\mathbf{X}^N) \to \tau_V(P) \qquad ??$$

# Outline

# Asymptotics of Matrix models

One matrix models

Multi-matrix models

# One matrix models and Coulomb gases

If $X^N$ is a $N \times N$ Hermitian matrix with distribution

$$d\mathbb{P}_N^V(X^N) = \frac{1}{Z_N^V} e^{-N\mathrm{Tr}V(X^N)} dX^N$$

*Then* $X^N =_d U\mathrm{diag}(\lambda)U^*$

where

- $U$ follows the Haar measure on $U(N)$,
- $\mathrm{diag}(\lambda)$ is a diagonal matrix with entries following the Coulomb gas distribution

$$dQ_N^V(\lambda) = \frac{1}{Z_N^V} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} \prod_{1 \le i \le N} d\lambda_i$$

with $\beta = 2$.

# Large deviations and convergence

$$dQ_N^V(\lambda) = \frac{1}{Z_N^V} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^\beta e^{-N \sum_{i=1}^N V(\lambda_i)} \prod_{1 \le i \le N} d\lambda_i$$

Theorem (Voiculescu '93, Ben Arous-G '97, Garcia-Zelada '19)

*Assume $V(x) \ge (\beta + \epsilon) \ln |x| + C$, $V$ continuous. The law of $\hat{\mu}_N = \frac{1}{N} \sum \delta_{\lambda_i}$ satisfies a large deviations principle with speed $N^2$ and good rate function $\mathcal{E}_V(\mu) = J_V(\mu) - \inf J_V$ where*

$$J_V(\mu) = \frac{1}{2} \int \int (V(x) + V(y) - \beta \ln |x - y|) d\mu(x) d\mu(y)$$

*In other words* $Q_N^V(\hat{\mu}_N \simeq \mu) \simeq e^{-N^2(\mathcal{E}_V(\mu))}$.

*$\mathcal{E}_V$ achieves its minimum value at a unique probability measure $\mu_V$ towards which $\hat{\mu}_N$ converges almost surely.*

# Equilibrium measure

The equilibrium measure minimizes

$$J_V(\mu) = \frac{1}{2}\int\int (V(x) + V(y) - \beta \ln|x-y|) d\mu(x) d\mu(y)$$

It is the unique probability measure such that there exists a constant $C$ so that

$$V(x) - \beta \int \ln|x-y| d\mu(y) \geq C \quad a.s$$

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with equality $\mu$ almost surely.It implies that $\mu$ satisfies the limiting Dyson-Schwinger equations : for any $f \in C_b^1$

$$\frac{\beta}{2}\int \frac{f(x)-f(y)}{x-y} d\mu(x) d\mu(y) = \int f(x) V'(x) d\mu(y)$$

If $V$ has deep wells, solutions have a disconnect support localized around the minimizers of $V$ and the DS equations have a solution for each choice of masses of these connected pieces of the support.

# Global fluctuations

$$dQ_N^V(\lambda) = \frac{1}{Z_N^V} \prod_{1\le i<j\le N} |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} \prod_{1\le i\le N} d\lambda_i$$

## Theorem (Johansson 97, Borot-G 13, Shcherbina 13)

*Assume $V$ smooth enough and the density of $\mu_V$ vanishes like a square root at its boundary then*

- *If the support of $\mu_V$ is connected, then for smooth enough test functions*

$$N \int f(x) d(\hat{\mu}_N - \mu_V)(x) \Rightarrow N(m_f^V, \sigma_f^V).$$

- *If $supp(\mu_V) = \cup_{1\le i\le n}[a_i, b_i]$, $b_{i-1} < a_i < b_i$, the number of eigenvalues in $[a_i, b_i]$ fluctuates like a discrete Gaussian (with mean which may not converge) and, conditionally to these filling fractions, the above holds up to proper recentering.*

# Idea of the proof : Dyson-Schwinger equations

**Principle :** "Moments of $\hat{\mu}_N$ satisfy equations that can be asymptotically solved"

**Example :** Let $f$ be a smooth function,

$$\mathbb{E}\left[\frac{\beta}{2}\int \frac{f(x)-f(y)}{x-y} d\hat{\mu}_N(x) d\hat{\mu}_N(y)\right] = \mathbb{E}\left[\int V'(x) f(x) d\hat{\mu}_N(x)\right]$$

$$+(\frac{\beta}{2}-1)\frac{1}{N}\mathbb{E}[\int f' d\hat{\mu}_N].$$

is a consequence of

$$\sum_{i=1}^{N} \int \partial_{\lambda_i}\left( f(\lambda_i) \frac{dQ_N^V}{d\lambda}(\lambda)\right) d\lambda = 0$$

**Rmk :** Another approach is to perform infinitesimal change of variables $\lambda_i \to \lambda_i + \frac{1}{N} f(\lambda_i)$ (cf Leblé-Serfaty/ Collins-G-Maurel Segala etc)

# Asymptotics of Matrix models

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Multi-matrix models

# Multi-matrix models

For which potential $V$ there exists $\tau_V$ s.t $\forall P \in \mathbb{C}\langle X_1, \ldots, X_d \rangle$

$$\frac{1}{Z_N^V} \int \frac{1}{N} \mathrm{Tr}\left(P(\mathbf{X}^N)\right) e^{-N \mathrm{Tr} V(\mathbf{X}^N)} d\mathbf{X}^N \to \tau_V(P) \qquad ??$$

- If $V(\mathbf{X}) = \sum V_i(X_i)$($\tau_V$ is the law of free variables with distribution $\mu_{V_i}$(Voiculescu '91)),

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- If $V(\mathbf{X}) = \sum V_i(X_i) + \epsilon W(\mathbf{X})$ with $V_i$ strictly convex and $\epsilon$ small (G-Maurel Segala '06 and Collins-G-MS '09),

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- If $V(\mathbf{X}) = V_1(X_1) + X_1 X_2 + V_2(X_2)$ (Mehta '81, Matytsin '97, G-Zeitouni '03, G-Huang '21)

But what can we say about the "unsolvable " commutator model

$$V_\beta(\mathbf{X}) = -\beta[X_1, X_2]^2 + V_1(X_1) + V_2(X_2)$$

# Strategy to study multi-matrix models

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## Strategy to study multi-matrix models

Large deviations are not well understood, see Voiculescu's entropies and Biane-Capitaine-G '03.

1. Show that the operator norm of the matrices stay bounded with large probability,
2. Deduce that the empirical distribution

$$\hat{\mu}^N(P) := \frac{1}{N}\mathrm{Tr}(P(\mathbf{X}^N))$$

   is tight,
3. Show that any limit point of $\hat{\mu}^N$ satisfies the Dyson-Schwinger equations,
4. Show that there exists a unique solution to this equation.

If $V$ is convex, the first point is deduced from Brascamp Lieb inequality. In perturbative situations, uniqueness follows by showing that uniqueness is stable under small perturbation (when $V$ stay convex) and in convex situations, from uniform convergence of the associated Langevin dynamics.

# Matrix models at low temperature(G- Maurel-Segala '22)

- There are sufficient conditions on $V$ such that $\max_i \|X_i^N\|_\infty$ stay bounded with overwhelming probability. This includes

$$V(\mathbf{X}) = \sum c_i X_i^{2D} + U(\mathbf{X})$$

with $c_i > 0$, $D \in \mathbb{N}^*$ and $U$ of degree bounded by $2D-1$.

- Under this condition, any limit point $\tau_V$ of $\hat{\mu}^N$ satisfies the Dyson-Schwinger equations.
- If $V = \beta V_0 + W$, there exists a finite $B$ such that for all $k$

$$\tau_V(|\mathcal{D}_i V_0|^{2k}) \leq (B/\beta)^k.$$

Kazakov-Zheng '21 : Relaxation Bootstrap method for the numerical solution of multi-matrix models. Conjecture : Additional symmetries give uniqueness of solutions to loop equations.

# Low temperature expansion ( G–Maurel Segala '22) : specific models

$$\mathbb{P}^N_{V_\beta}(d\mathbf{X}^N) = \frac{1}{Z^N_{V_\beta}} \exp\{-N\mathrm{Tr}(V_\beta(\mathbf{X}^N))\} d\mathbf{X}^N$$

and

$$\tau^N_{V_\beta}(P) = \int \frac{1}{N}\mathrm{Tr}(P(\mathbf{X}^N)) d\mathbb{P}^N_{V_\beta}(\mathbf{X}^N)$$

- If $V_\beta(\mathbf{X}) = \beta V(\mathbf{X}) + W(\mathbf{X})$ with $V$ minimum at a unique $m^*$ with $\mathrm{Hess}(\mathrm{Tr} V)(m^*) > 0$. Then for $\beta$ large enough, we are back to the convex situation.

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- If $V_\beta(\mathbf{X}) = \beta \sum V_i(X_i) + \sum_i Z_i(\mathbf{X})$,
  - $V_i$ minimum at $(x^i_j)_{1 \leq j \leq m_i}$ with $V_i(x) - V_i(x^i_j) \simeq c^i_j (x - x^i_j)^{2k^i_j}$,
  - $Z_i(X) = \prod (X_i - x^i_j) Q_i(\mathbf{X})$,

  $\tau^{V_\beta}_N$ converges towards $\tau_\beta$ for $\beta$ large. $\tau_\infty$ is the law of free variables with law $\frac{1}{\sum k^i_j} \sum_{j=1}^{m_i} k^i_j \delta_{x^i_j}$.

# The "unsolvable" Commutator model

Given by

$$\tau_{V_\beta}^N(P) = \int \frac{1}{N}\mathrm{Tr}(P(\mathbf{X}^N))d\mathbb{P}_{V_\beta}^N(\mathbf{X}^N)$$

with $V_\beta(\mathbf{X}) = -\beta[X_1, X_2]^2 + V_1(X_1) + V_2(X_2)$

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- If $V_i$ are quadratic, the model was studied by Kazakov, Kostov, Nekrasov '98.
- If $V_i$ minimum at $(x^i_j)_{1 \le j \le m_i}$, $V_i(x) - V_i(x^i_j) \simeq c^i_j(x - x^i_j)^2$, and $c^i_j > 0$. Then

$$\lim_{\beta\to\infty} \lim_{N\to\infty} \tau^N_{V_\beta}(P)$$

is the law of two commuting variables with laws

$$\sum_j \frac{(c^i_j)^{-1/2}}{\sum_k (c^i_k)^{-1/2}} \delta_{x^i_j}, i \in \{1, 2\}.$$

The proof uses fine large deviations estimates to fix the filling fractions, based on localisations close to the critical points.

# A key tool : estimates by transport.

## Lemma

*Let $f : \mathbb{R}^d \mapsto \mathbb{R}^+$ be a measurable function with $\int |f(x)|dx < \infty$. Let $d\mathbb{P}(x) = cf(x)dx$ be a probability measure on $\mathbb{R}^d$. If $\phi : A \to \mathbb{R}^d$ is a $\mathcal{C}^1$ diffeomorphism onto its image then*

$$\mathbb{P}(X \in A) \leqslant \sup_{x \in A} \frac{f(x)}{f \circ \phi(x) J_\phi(x)}$$

*where $J_\phi$ is the Jacobian of $\phi$ : $J_\phi(x) = \det(\partial_i \phi_j(x))$.*

Indeed

$$\begin{aligned} \mathbb{P}(X \in A) &= \int_{x \in A} \frac{f(x)}{f \circ \phi(x) J_\phi(x)} cf \circ \phi(x) J_\phi(x) dx \\ &\leq \sup_{x \in A} \frac{f(x)}{f \circ \phi(x) J_\phi(x)} \int_A cf \circ \phi(x) J_\phi(x) dx \end{aligned}$$

# A matrix inequality

## Lemma

*Let $\mathbf{X}_0^N$ be a $d$-tuple of $N \times N$ matrices and $\mathbf{X}_t^N$ solution of $\partial_t (X_i^N)_t = -g_i(\mathbf{X}_t^N)$ with $\mathbf{X}_0^N = \mathbf{X}^N$ . If $g(\mathbf{X}) = X_{i_1} \cdots X_{i_k}$, set $\partial_i g(X) = \sum_{j: i_j = i} X_{i_1} \cdots X_{i_{j-1}} \otimes X_{i_{j+1}} \cdots X_{i_k}$, $\mathcal{D} g(X) = m(\partial g(X))$ with $m(A \otimes B) = BA$. Then*

$$\mathbb{P}_N^V(\mathbf{X}^N \in A) \leqslant e^{-\inf_{\mathbf{X}_0^N \in A} \{\int_0^t \sum_i (\mathrm{Tr} \otimes \mathrm{Tr}(\partial_i g(\mathbf{X}_s^N)) - N \mathrm{Tr} \mathcal{D}_i V(\mathbf{X}_s^N) g(X_s^N)) ds)\}}.$$

As a consequence

- If $\tau_V$ is a limit point of $\frac{1}{N}\mathrm{Tr}(P(\mathbf{X}^N))$ it satisfies the limiting Dyson-Schwinger equations : for all $i \in \{1, \ldots, d\}, all g smooth$

$$\tau_V \otimes \tau_V(\partial_i g) = \tau_V(\mathcal{D}_i V g)$$

- The norm $\max_i \|X_i\|$ can be bounded if $V$ such that there exists $\eta > 0$ and $A$ finite so that

$$\mathrm{Tr}(\sum_{j=1}^{d} (X_j^N)^{2k+1} \mathcal{D}_j V(\mathbf{X}^N)) \geqslant \mathrm{Tr}(\eta \sum_{j=1}^{d} (X_j^N)^{2(k+1)} - A \sum_{j=1}^{d} (X_j^N)^{2k}).$$

# Open Questions

- Convergence of multi-matrix models beyond perturbative or convex cases is open in general. Investigate the phase transition ?Only convergence would have important consequences in entropy theory in free probability.

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- Convergence of multi-matrix models beyond perturbative or convex cases is open in general. Investigate the phase transition ?Only convergence would have important consequences in entropy theory in free probability.
- Following Kazakov-Zheng, find natural symmetry conditions to insure uniqueness ?
- Fluctuations of multi-matrix models are known in perturbative situations. What in general ?
- Understand the commutator model in the large (but not infinite) $\beta$ case, and in general ?