Coulomb gas vs. Laughlin states on Riemann surfaces

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Coulomb gas and universality, December 5-9, 2022

based on pre-covid work and on recent work with Dimitri Zvonkine (CNRS and Paris-Versailles U.

Coulomb gas partition function for N particles on a surface

$$\mathcal{Z} = \int_{\Sigma^N} e^{-q\sum_{n<m}^{N} G(z_n, z_m) - \sum_{n=1}^{N} V(z_n)} \prod_{n=1}^{N} d^2 z_n \qquad q \in \mathbb{R}_+$$

V is an external potential $V : \Sigma \to \mathbb{R}$, G is the Green function of the 2d Laplace-Beltrami operator

When $\Sigma = \mathbb{C}$, $G(x,y) = -\ln|x-y|$

Laughlin state on C^N is a wave function whose L2 norm equals Z when $q \in \mathbb{N}^*$ and $V = \frac{B}{2}|z|^2$

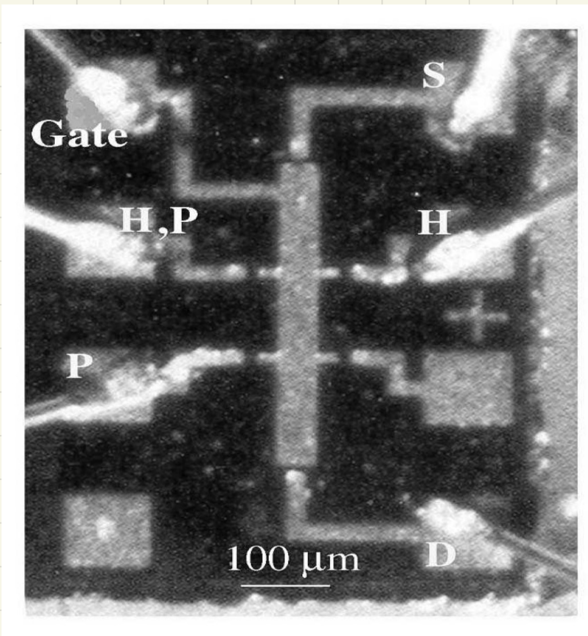
$$\Psi : \mathbb{C}^N \to \mathbb{C}$$

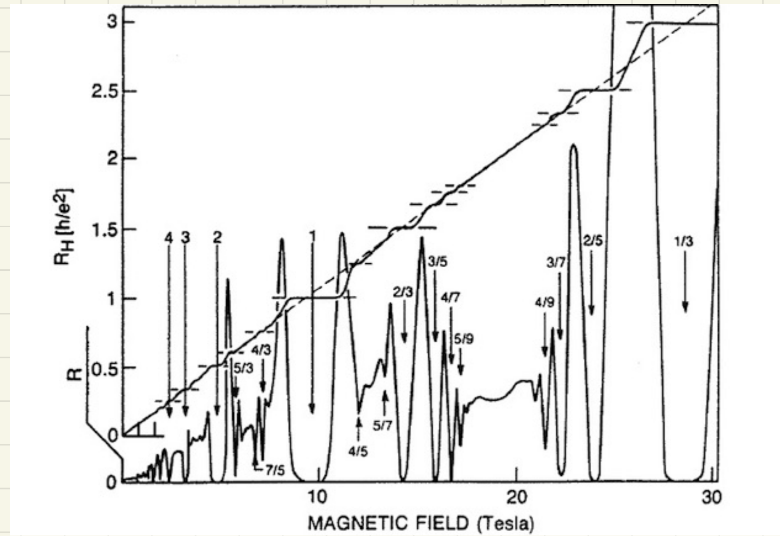
$$\Psi(z_1, \dots, z_N) = \prod_{n<m}^{N} (z_n - z_m)^q \, e^{-\frac{B}{4}\sum_{n=1}^{N} |z_n|^2} , \qquad B > 0 \text{ magnetic field}$$

$$\mathcal{Z} = \int_{\mathbb{C}^N} |\Psi|^2 \prod_{n=1}^{N} d^2 z_n$$

Quantum Hall effect

Quantum Hall states are discovered in early 80’s in GaAs semiconductors at low temperatures and high perpendicular magnetic fields. Fractional QHE states are particularly interesting because strongly-interacting.





Hall conductance is quantized

$$\sigma_H = \frac{1}{R_H} = \frac{p}{q} \cdot \frac{h}{e^2} \quad , \quad p, q \in \mathbb{N}$$

The hamiltonian operator for N interacting electrons in the (complex) plane and in perpendicular magnetic field has the form

$$\hat{H} = \sum_{n=1}^{N} D_n^{+} D_n + \sum_{n=1}^{N} \sum_{i=1}^{Q} V_{w_i}(z_n) + e^2 \sum_{n<m}^{N} V(z_n, z_m)$$

impurities

Coulomb interaction

$$D_n = \frac{\partial}{\partial \bar{z}_n} + \frac{B}{2} z_n$$

hence the kinetic term vanishes identically for holomorphic functions of electron coordinates

$$\Psi = f(z_1, \ldots, z_N)\, e^{-\frac{B}{2} \sum_{n=1}^{N} |z_n|^2}$$

lowest Landau level

How to minimize the interaction term?

Laughlin' paradigm '83: the following is the good Hilbert space on the plateaux of QHE

$$\mathcal{H}_{N,d,q} = \left\{ \Psi \in L^2(\mathbb{C}^N) \,\middle|\, \Psi = P(z_1, \dots, z_N)\, e^{-\frac{B}{2}\sum_{n=1}^{N} |z_n|^2} \right\}$$

* $P \in \mathbb{C}[z_1, \dots, z_N]$ are symmetric or anti-symmetric

degree d polynomials in N letters z_n. In physics integer d is denoted as N_ϕ for magnetic flux.

* For the plateau 1/q polynomials vanish to the order q when any two particles meet, i.e.

$$P \in \bigcap_{n \neq m} (z_n - z_m)^q$$

* different plateaux p/q correspond to other « interaction ideals »

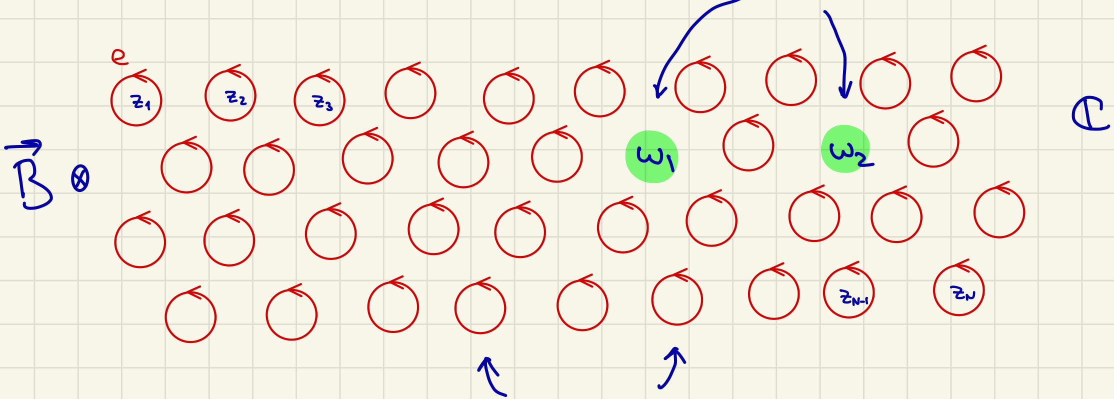
$$\tilde{P} \in \bigcap_{n \neq m \neq k} (z_n - z_m, z_m - z_k)$$

Moore-Read pfaffian state

* Symmetric polynomial can have Q prescribed zeroes (Q localized quasiholes), placed e.g. at impurities

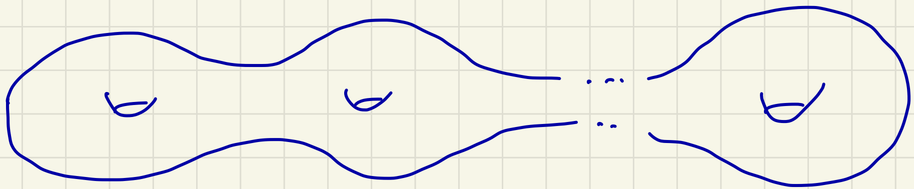
$$\Psi_L = \prod_{i=1}^{Q} \prod_{n=1}^{N} (z_n - w_i) \cdot \prod_{n<m}^{N} (z_n - z_m)^q$$

quasiholes (minimize impurities potential)



electrons on the lowest Landau level

Higher genus surfaces (Haldane-Rezayi, Wen-Niu, Avron-Seiler-Zograf, late 80s- early 90s)



Conjecture 1. What is the maximal number of particles that a Laughlin state can contain? (“completely filled state”)

$$N = \frac{1}{q} d + 1 - g$$ on a surface of genus g (Wen-Zee formula)

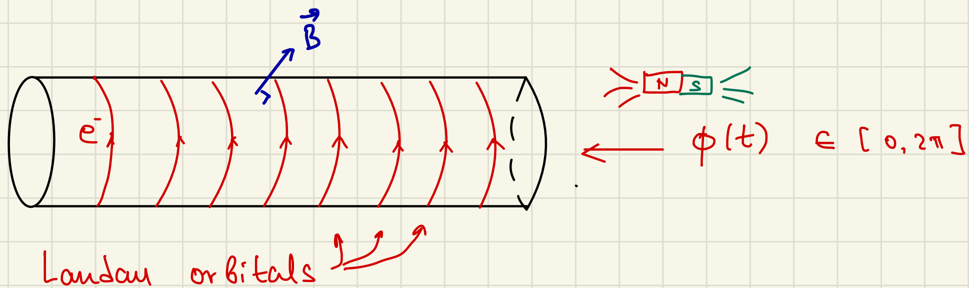
Conjecture 2. How many states there are for the maximal number of particles?

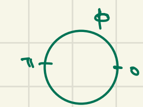
q^g dimensional vector space of Laughlin states (Wen-Niu conjecture)

Note that the last formula is independent of number of particles N and total magnetic flux d

(« topological degeneracy »)

So far the state is static, where are the currents? Laughlin again: To make electrons move, let's wrap the sample into a cylinder and thread a magnetic field through the hole. This brings the parameter space M of Aharonov-Bohm (solenoid) phases into the game.





"parameter space" S^1

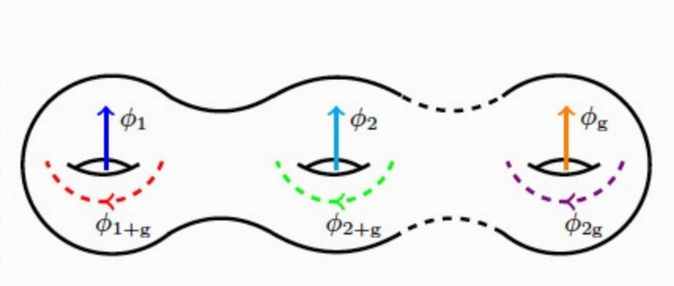
"Riemann surface"

As ϕ changes from 0 to 2π, and due to the Faraday's law and Lorentz force, one electrons drops from the left edge and another one appears on the right edge.

Hence Hall conductance equals one.

Consider Laughlin states on a genus-g surface and thread magnetic fields (=Aharonov-Bohm phases=solenoid phases) through the 2g holes of the surface. $\{\phi_a\}_{a=1\ldots 2g} \in (S^1)^{2g}$

Laughlin states form a vector bundle E over the parameter which is a 2g- torus (known also as the Jacobian variety of the Riemann surface).



Conjecture 3. (Avron-Seiler'85, +Zograf'95)

Hall conductance $\quad \sigma_H \in \dfrac{c_1(E)}{rk(E)}$

Changing AB flux through the b-th hole $\phi_b = V_b t$ creates a Hall current though the a-th hole $I_a = (\sigma_H)_{ab} V_b$

Metaconjecture 4. (Wen,Read etc…) Vector bundles of « good » FQHE states over appropriate parameter spaces* are « projectively flat », maybe asymptotically at large N.

*examples parameter spaces considered in QHE: AB phases, moduli spaces of surfaces, moduli spaces of quasiholes and quasiparticles….

Laughlin state on a Riemann surface

The definition of the Laughlin states on a Riemann surface is analogous to the one given before for the complex plane, except that we replace the notion of holomorphic polynomials of degree d by the notion of sections of holomorphic line bundle of degree d (which is the flux of magnetic field B).

Def. Let $L \to \Sigma$ be a holomorphic line bundle of degree $d \geq 2g-1$. Take $N \geq g$ and fix a positive integer q. Consider Nth power Σ^N and let $z_1, \dots, z_N$ be N coordinates. Denote by $\pi_1, \dots, \pi_N$ the N projections from Σ^N to Σ

Consider the following line bundle on Σ^N

$$L^{\boxtimes N} = \pi_1^* L \otimes \cdots \otimes \pi_N^* L$$

The Laughlin state of weight q is a section ψ of $L^{\boxtimes N}$, which is

* completely symmetric or anti-symmetric for q even, resp. odd
* vanishes to the order q on all the diagonals $\Delta_{nm} = \{ z_n = z_m \}$

Let us denote the vector space of Laughlin states as $\mathcal{H}_{N,d,q,g}(L)$ for a given bundle L

The parameter space M will be the moduli space of degree-d line bundles L, the Picard variety $\mathrm{Pic}^d(\Sigma)$ which is a g-dimensional complex torus (isomorphic to Jacobian $\mathrm{Jac}(\Sigma)$). This is the space of inequivalent configurations of the magnetic field of given flux d piercing the surface, parametrized by the solenoid phases through the 2g holes on the surface

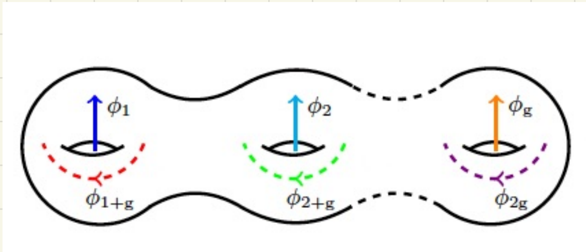
$$\{\phi_a\}_{a=1,\dots,2g} \in [0,2\pi]^{2g}$$

Varying L, by varying the solenoid fluxes we obtain the vector bundle of Laughlin states

$$\pi : E \to M$$

$$M = \mathrm{Jac}(\Sigma)$$

$$\pi^{-1}(\{\phi\}) = \mathcal{H}_{N,d,q,g}(L_\phi)$$



Thm 1. (Zvonkine, SK 21)

$$\boxed{\dim \mathcal{H}_{N,d,q,g} = \sum_{k=0}^{g} \binom{g}{k} \binom{N-g+p}{k-g+p} q^k}$$

Notation $\quad p = d - q(N+g-1) \qquad \binom{a}{b} = \frac{a!}{b!(a-b)!}$

Corrollary 1. there are no Laughlin states with p<0, in other words the for given d,q,g

the maximal number of particles (completely filled state) is given by

$$N_{max} = \left[\frac{d}{q}\right] + 1 - g$$

Wen-Zee formula

Corrollary 2. if moreover q divides d, $\quad \dim \mathcal{H}_{N_{max},d,q,g} = q^g$

Wen-Niu conjecture

Quantum optimal packing problem — how many configurations of the most dense (« completely filled ») state?

Laughlin state on a Riemann surface

Why not write

$$\left| \Psi(z_1, \dots, z_N) \right|^2 = e^{-q \sum_{n<m}^{N} G(z_n, z_m)}$$

When q=1, Laughlin state is supposed to be the Slater determinant

$$\Psi = \det s_i(z_j) \Big|_{i,j=1,\dots,N}$$

where s_i is a basis of holomorphic sections of the magnetic line bundle L.

This one has N-1+g zeroes in each variable (e.g. in z_1) by the Riemann-Roch theorem.

The one above has only N-1 zeroes.

Thm 2. (SK19) In the completely filled state here is the basis of Laughlin states

$$\boxed{|\Psi_i|^2 = \left|\Theta\begin{bmatrix} i/q \\ 0 \end{bmatrix}\left(q\sum_{n=1}^{N} z_n - D + \phi\right)\right|^2 e^{-q\sum_{n<m}^{N} G(z_n, z_m)}}$$

$i = (1, \dots, q)^g$ $\qquad$ $\phi \in M = \mathrm{Jac}(\Sigma)$

D is a divisor (vanishing set of Ψ),

encodes magnetic line bundle L

Θ is a basis of level-q Riemann theta functions

$$\Theta(e, \tau) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (n, \tau n) + 2\pi i (n, e)}$$

$e \in \mathbb{C}^g/\Lambda$, $\Lambda = \{ m'_j + \tau_{jl} m_l \mid m, m' \in \mathbb{Z}^g \}$

$\mathrm{Im}\,\tau > 0$

τ period matrix of Σ, $\tau_{jl} = \int_{b_j} \omega_l$, ω_l is a basis of holomorphic 1-forms.

Abel map: $\Sigma^N \xrightarrow{f} \mathbb{C}^g/\Lambda$

$(z_1, \dots, z_N) \to \sum_{n=1}^{N} \int_{z_0}^{z_n} \omega_j$ $\qquad$ $f_* \Theta$

Ψ_i give a local frame field of the Hermitian vector bundle E of Laughlin states over Jacobian, so by the standard formula in differential geometry of vector bundles, the L2 overlaps of the frames determine curvature of the bundle

$$H_{ij}(\phi) = \langle \Psi_i , \Psi_j \rangle_{L^2(\Sigma^N)} \qquad i,j = 1, \dots, rk(E)$$

$$R = \bar{\partial}_\phi \left(\partial_\phi H \cdot H^{-1} \right)$$

H_{ij} are Coulomb-gas type integrals, which (hopefully some day) could be tackled asymptotically at large N. If H is a scalar matrix, then so is R, in this case the bundle is projectively flat.

$$R_{ij} = \alpha \, I_{ij} \qquad \alpha \in \Omega^2(M)$$

Testing projective flatness

Once we pick a connection on a rank-r complex vector bundle and compute the curvature, $R \in \Omega^2(\mathrm{End}(E))$

its trace is a closed differential two-form on M, which defines a cohomology class on M called

first Chern class.

$$c_1(E) = \frac{i}{2\pi} \, \mathrm{tr}\, R \in H^2(M, \mathbb{R})$$

Now, the traces of higher powers of the curvature also define closed differential forms on M,

their curvatures are called Chern characters

$$ch_m(E) = \left(\frac{i}{2\pi}\right)^m \frac{1}{m!} \, \mathrm{tr}\, R^m \in H^{2m}(M, \mathbb{R})$$

The key fact for us is the following — if E is projectively flat,

then this relation holds in cohomology, i.e. up to a total derivative.

$$\boxed{ch_m(E) = \frac{[c_1(E)]^m}{m!\, r^{m-1}}}$$

Thm 3 (Zvonkine, SK21)

$$\boxed{ch_m(E) = \sum_{k=m}^{g} \binom{g-m}{k-m} \binom{N-g+p}{k-g+p} q^{k-m} \frac{\Theta^m}{m!}} \qquad (m \le g)$$

The 2-form $\Theta = \sum_{a=1}^{g} d\phi_a \wedge d\phi_{a+g}$ is the theta class.

Corollary: in the completely filled state (p=0) the first Chern class is $c_1(E) = q^{g-1} \cdot \Theta$

and the rank (E) is q^g

$$\sigma_H \in \frac{c_1(E)}{rk(E)} = \frac{1}{q}\,\Theta \qquad \text{(Avron et al formula)}$$

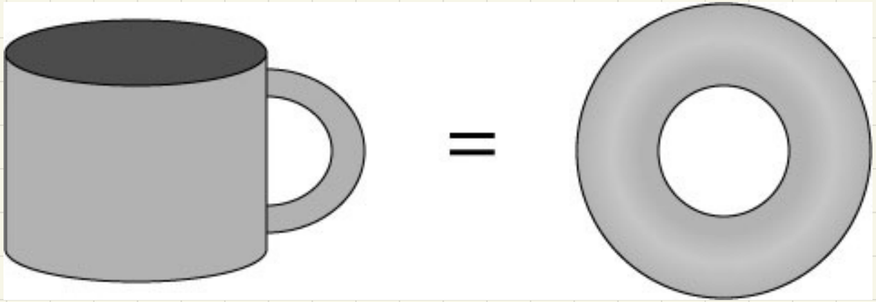
Idea of the proof: use Hirzebruch-Riemann-Roch theorem to compute the rank and Grothendieck-Riemann-Roch formula to compute the Chern classes

Application: topological states of matter

It is often said that FQHE states are first examples of “topological states of matter”.

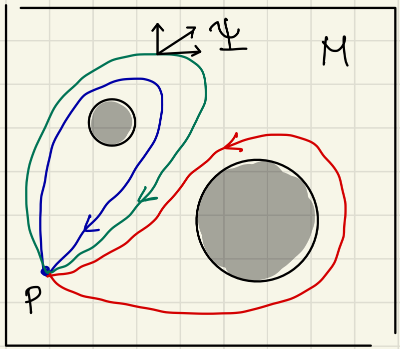
What does it mean for the state of matter to be “topological”?

Topology is concerned with geometric properties that are preserved under continuous deformations.



Definition: adiabatic transport along a path in the parameter space is independent under continuous deformations of the path. This means the bundle is flat — locally on M the bundle looks like a product space of M and C^r with constant transition functions between the local charts.

Remark: usually all this assumes existence of a gap.



For a complex vector bundle E —> M. the following is equivalent:

1. E is flat

2. E admits a flat connection $\nabla : \Gamma(E) \to \Omega^1(M, \Gamma(E))$

$R = \nabla^2 = 0$

3. E is defined by a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$

(Kobayashi, *Complex vector bundles)*

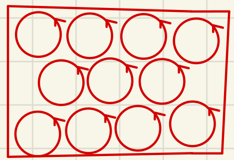
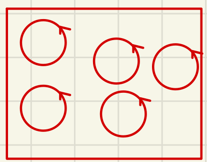
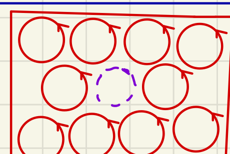
Fundamental group $\pi_1(M)$ is the set of all loops modulo continuous deformations between them, i.e. on the picture above it would not distinguish between the blue and green loops, but distinguish them from the red one.

Since we are in quantum mechanics, we will allow for transport independent of the continuous deformations of the path "up to a phase factor" — i. e. projectively flat bundles. For projectively flat bundles there should exist a connection ∇ with curvature being a scalar matrix

$$R = \alpha I_r \quad , \quad \alpha \in \Omega^2(M)$$

It is hard to construct the projectively flat connection, but we can test the topological state of matter by whether it satisfies the projective flatness test (« geometric test »)

$$\boxed{\mathrm{ch}_m(E) = \frac{[c_1(E)]^m}{m!\, r^{m-1}}}$$

Laughlin state		geometric test $\boxed{\mathrm{ch}_m(E) = \frac{[c_1(E)]^m}{m!\, r^{m-1}}}$?	
Completely filled		✓	
Partially filled		✗	independent 2nd and higher Chern classes
Localized quasi-holes		✓	

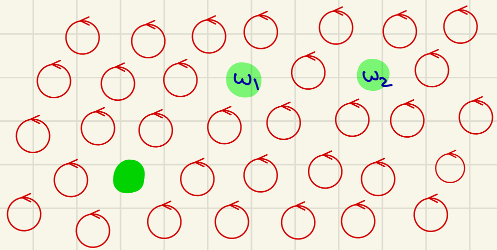
Another application : Hall conductance and plateaux

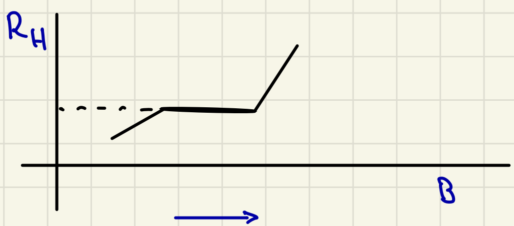
* In a completely filled state the Hall conductance is quantized

$$\boxed{\sigma_H = \frac{1}{q} \; \Theta}$$

* Once p quasiholes become non-localized (e.g. no impurities left), the Hall conductance starts to deviate from the quantized value

$$\boxed{\sigma_H = \left(\frac{1}{q} - \frac{p}{g\, q^2 N} + O\left(\frac{1}{N^2}\right) \right) \Theta}$$





increasing degree d

* We computed rank and Chern classes of the bundle of Laughlin states over the Jacobian, proving Wen-Zee formula and Wen-Niu topological degeneracy.

* In order to prove projective flatness we still need to find a connection $\nabla : \Gamma(E) \to \Omega^1(M, \Gamma(E))$ whose curvature is a scalar matrix

$$R_\nabla = \alpha I_E \qquad \text{for some} \qquad \alpha \in H^2(M)$$

* First Chern class is the quantized Hall conductance. What are the signatures of the second Chern class, if any?

* Apply geometric test in other situations — other FQHE plateaux, Brillouin zone, etc. Prerequisites: degenerate states, gap, parameter space of complex dimension at least two.

Thank you!

Editors' Suggestion

Geometric Test for Topological States of Matter

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(Received 31 May 2021; revised 1 September 2021; accepted 21 December 2021)

We generalize the flux insertion argument due to Laughlin, Niu-Thouless-Tao-Wu, and Avron-Seiler-Zograf to the case of fractional quantum Hall states on a higher-genus surface. We propose this setting as a test to characterize the robustness, or topologicity, of the quantum state of matter and apply our test to the Laughlin states. Laughlin states form a vector bundle, the Laughlin bundle, over the Jacobian—the space of Aharonov-Bohm fluxes through the holes of the surface. The rank of the Laughlin bundle is the degeneracy of Laughlin states or, in the presence of quasiholes, the dimension of the corresponding full many-body Hilbert space; its slope, which is the first Chern class divided by the rank, is the Hall conductance. We compute the rank and all the Chern classes of Laughlin bundles for any genus and any number of quasiholes, settling, in particular, the Wen-Niu conjecture. Then we show that Laughlin bundles with nonlocalized quasiholes are not projectively flat and that the Hall current is precisely quantized only for the states with localized quasiholes. Hence our test distinguishes these states from the full many-body Hilbert space.