Scaling limits of Gaussian β-ensembles characteristic polynomial

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1 Hermite polynomials $\boxed{\beta = \infty}$

Let $(H_k)_{k=0}^{\infty}$ be the orthonormal polynomials w.r.t. $x \to \sqrt{\frac{2N}{\pi}} e^{-2Nx^2}$ on $\mathbb{R}$.

- $(H_k)_{k=0}^{\infty}$ satisfies a recurrence relation:

$$\begin{cases} x H_k(x) = \sqrt{\frac{k+1}{4N}} H_{k+1}(x) + \sqrt{\frac{k}{4N}} H_{k-1}(x) \\ H_0 = 1 \end{cases}$$

- $$x \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ \vdots \\ \vdots \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{J} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ \vdots \\ \vdots \end{pmatrix}$$

$$\boxed{a_k = \sqrt{\frac{k}{4N}}, \quad b_k = 0.}$$

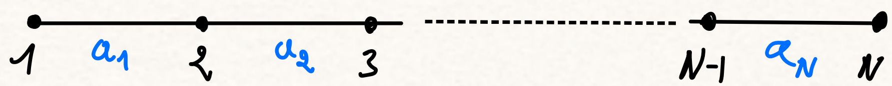
- $H_k(x) = \left(\prod_{j=1}^{k} a_j^{-1} \right) \det \left(x - J_k \right)$ for $k \in \mathbb{N}_0$.

<u>Thm</u> Let $(\gamma_1, \dots, \gamma_k)$ be the zeros of H_k.
The eigenvalues of J_k are $(\gamma_1, \dots, \gamma_k)$ and the eigenvectors are

$$\begin{pmatrix} H_0(\gamma_1) & \cdots & H_0(\gamma_k) \\ \vdots & & \vdots \\ H_{k-1}(\gamma_1) & \cdots & H_{k-1}(\gamma_k) \end{pmatrix}.$$

$a_{Nt} \sim \sqrt{t}/2$ for $t \in [0,1]$

- Graph associated to J_N:



$$\mathrm{Tr}\left(J_N^{\alpha} \right) \sim N \int_0^1 \binom{\alpha}{\alpha/2} \left(\sqrt{t}/2 \right)^{\alpha} dt \quad \text{for any } \alpha \in \mathbb{N}.$$

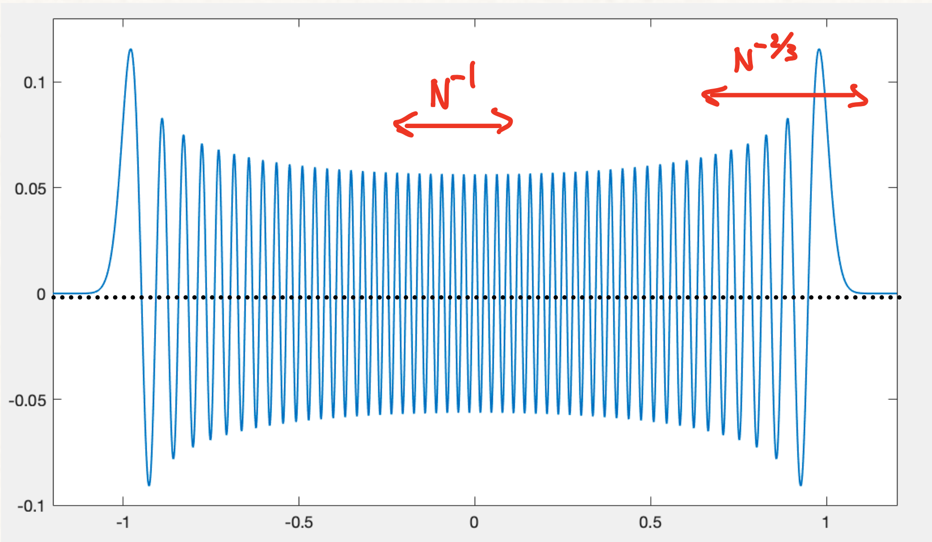
$$\mathrm{Tr}\left(J_N^{\alpha}\right) \sim N \frac{\binom{\alpha}{\alpha/2}}{\alpha+1} \qquad \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j} \longrightarrow \text{semicircle law } \rho \text{ on } [-1,1]$$

<u>Thm</u> [Stieltjes] Define $\boxed{\mathcal{E}_N(\lambda) := \sum_{i<j} \log|\lambda_i - \lambda_j|^{-1} + N\sum_{i=1}^{N} \lambda_i^2}$.

The minimizer of H_N is attained at $(\gamma_1, \dots, \gamma_N)$.

Define the Gibbs measure $\mathbb{P}_N^{\beta}[d\lambda] = Z_N(\beta)^{-1} e^{-\beta \mathcal{E}_N(\lambda)}$ on $\mathbb{R}^N$.

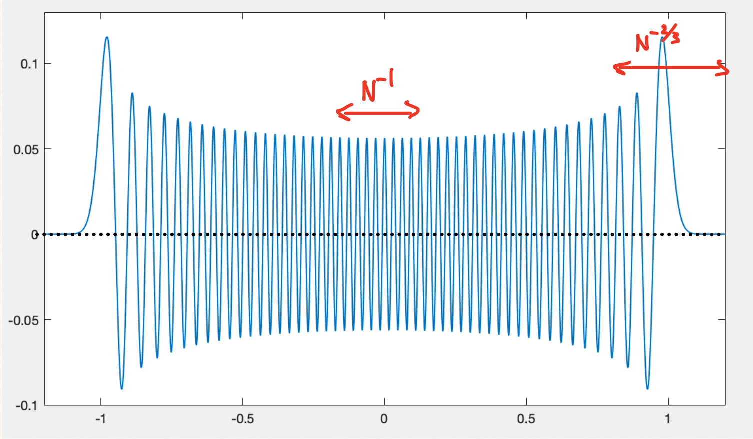
$\mathbb{P}_N^{\beta}$ concentrates around $(\gamma_1, \dots, \gamma_N)$ as $\beta \to \infty$.



Plot of $x \to H_N(x)\, e^{-Nx^2}$.

$N = 200$

Plancherel - Rotach asymptotics



Let $\boxed{\phi_k(x) = H_k(x)\left(\frac{2N}{\pi}\right)^{1/4} e^{-Nx^2}}$

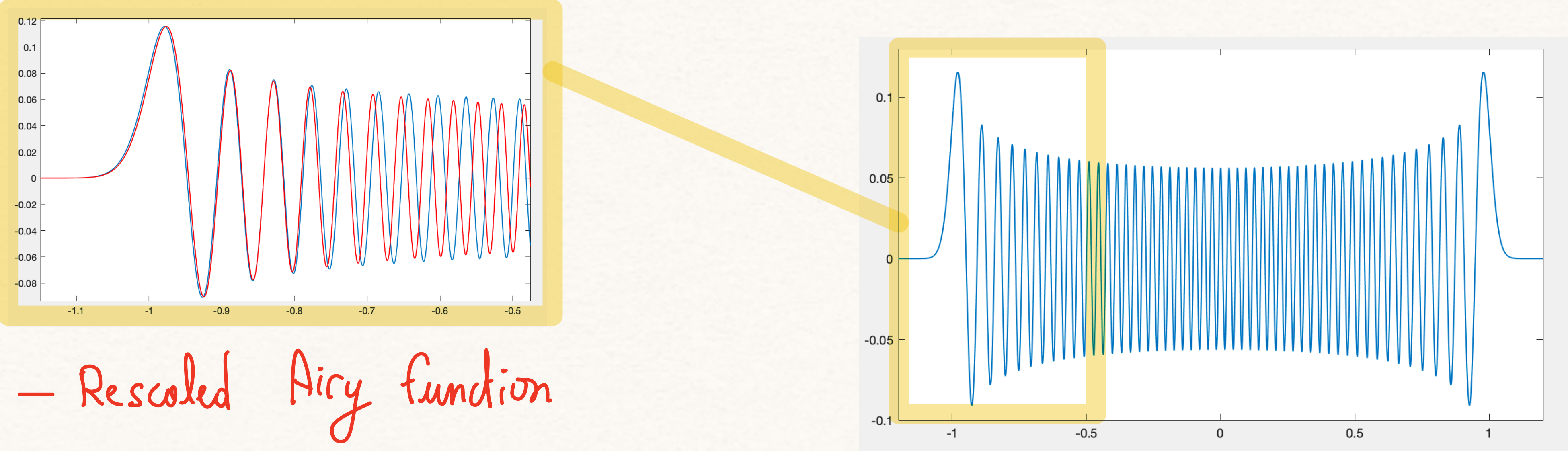
- Exponential small for $x \in \mathbb{R}\setminus[-1,1]$.

$$\phi_N(x) \simeq \frac{e^{-N\xi(x)}}{\sqrt{\pi}\,(x^2-1)^{1/4}} \qquad \xi(x) = 2\int_1^{|x|}\sqrt{u^2-1}\,du.$$

- Edge → Airy function: $\phi_N\left(-1+\frac{\lambda}{2N^{2/3}}\right) \simeq c\,N^{1/6}\underbrace{Ai(-\lambda)}_{\text{solution in } L^2(\mathbb{R}_+) \text{ of } -\partial_{tt}\phi + t\phi = 0}$

- Bulk for $x \in (-1,1)$.

$$\phi_N\left(x+\frac{\lambda}{N\rho(x)}\right) \simeq \frac{\cos\left(\pi N F(x) - \frac{\theta(x)}{2} + \pi\lambda\right)}{\sqrt{\pi/2}\,(1-x^2)^{1/4}} \qquad F(x) := \int_x^1 \frac{2}{\pi}\sqrt{1-u^2}\,du.$$

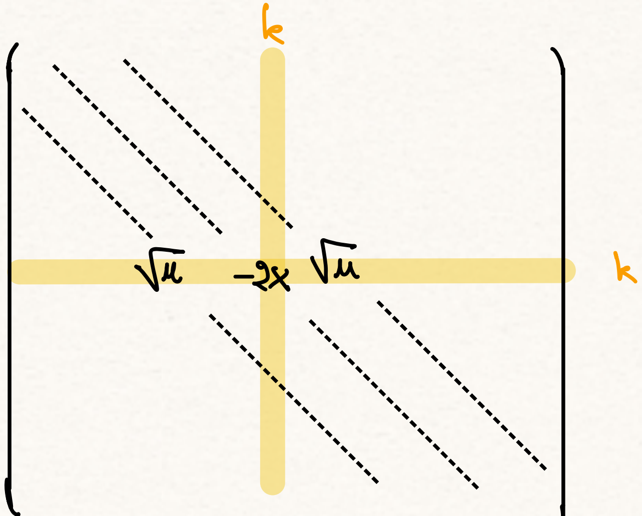


Turning point:

$$\Phi_k(x) \propto \det(x - J_k)$$

Transition if $\boxed{\mu = x^2.}$

$J - xI =$



$\mu = \frac{k}{N} \in [0,1]$

Transition window:

$$x = 1 + \frac{\lambda N^{-2/3}}{2} \qquad \lambda \in \mathbb{R}$$

$$k = N - t N^{1/3} \qquad t \in \mathbb{R}_+$$

$$N^{2/3} \times \begin{bmatrix} \ddots & \ddots & \ddots \\ \sqrt{k/N} & -2x & \sqrt{k/N} \\ \ddots & \ddots & \ddots \end{bmatrix} \simeq \partial_{tt} - t - \lambda$$

$$\Longrightarrow \quad N^{-1/6}\, \phi_{N - tN^{1/3}}\left(1 + \frac{\lambda}{2N^{2/3}}\right) \simeq c\, \mathrm{Ai}(t + \lambda) \qquad \text{as} \quad N \to \infty$$

Transfert matrices:

$$\begin{pmatrix} \phi_{k+1} \\ \phi_k \end{pmatrix} = \underbrace{\begin{pmatrix} x & -\frac{\sqrt{t}}{2} \\ \frac{2}{\sqrt{t}} & 0 \end{pmatrix}}_{T_k} \begin{pmatrix} \phi_k \\ \phi_{k-1} \end{pmatrix} \qquad t = \frac{k}{N}$$

$$\lambda_\pm = \lambda_\pm\left(\frac{x}{\sqrt{t}}\right)$$

$$T_k = \begin{pmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}\begin{pmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\lambda_\pm(u) = \begin{cases} u \pm \sqrt{u^2-1} \ ; & \underline{u>1} \\ e^{i\pm\theta(u)} \ ; \ \theta(u) = \arccos(u) \ ; & \underline{u<1} \end{cases}$$

Hyperbolic regime:

$$T_k \cdots T_1 \simeq \begin{pmatrix} \lambda_+^{(k)} & \lambda_-^{(k)} \\ 1 & 1 \end{pmatrix} \prod_{j=1}^{k} \lambda_+^{(j)} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } k \lesssim Nx^2$$

$$\Longrightarrow \quad \phi_N(x) \propto \phi_0 \prod_{j=1}^{N} \lambda_+\left(\frac{x}{\sqrt{j/N}}\right) \propto \exp\left(-N\xi(x)\right) \quad \text{if } x^2>1$$

Elliptic regime:

$$\begin{pmatrix} \xi_k \\ \overline{\xi}_k \end{pmatrix} := \begin{pmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \phi_{k+1} \\ \phi_k \end{pmatrix}$$

$$\boxed{\phi_k = 2\,\mathrm{Re}\,\xi_k}$$

$$\boxed{\xi_N \propto e^{i\vartheta_N}} \quad \text{where} \quad \vartheta_N = \theta\left(\frac{1}{\sqrt{N}}\right) + \cdots + \theta(x) \simeq \underbrace{N\int_{x^2}^{1} \theta\left(\frac{x}{\sqrt{t}}\right) dt}_{= \pi F(x)} - \frac{\theta(x)}{2}$$

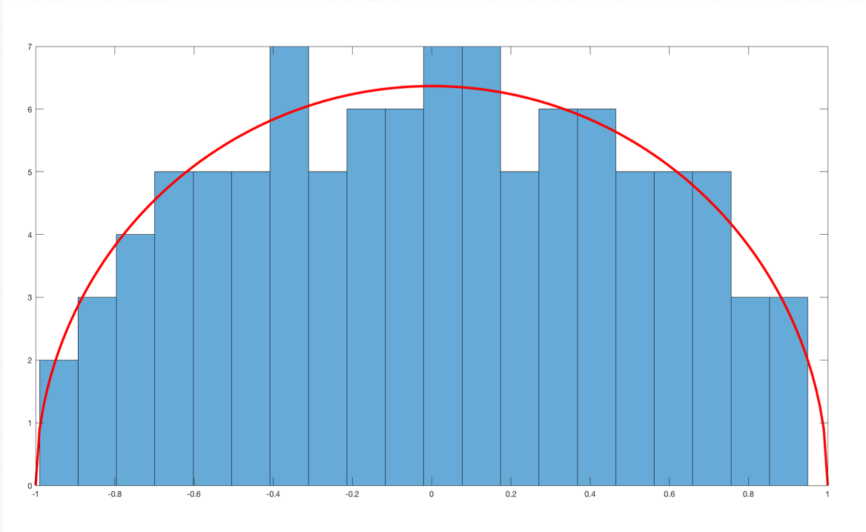
$$\Longrightarrow \quad \phi_k(x) \propto \cos\left(N\pi F(x) - \frac{\theta(x)}{2}\right) \quad \text{if } x^2 < 1 .$$

② Gaussian β- ensembles

$$\mathbb{P}_N^{\beta} = Z_N(\beta)^{-1} e^{-\beta \mathcal{E}_N} \qquad \mathcal{E}_N(\lambda) := \sum_{i<j} \log|\lambda_i - \lambda_j|^{-1} + N\sum_{i=1}^{N} \lambda_i^2$$

$\longrightarrow$ random configuration $\lambda_1 < \lambda_2 < \cdots < \lambda_N$.

Thm: With overwhelming probability



$$\frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j} \longrightarrow \rho$$

$$\lambda_k = \gamma_k + \mathcal{O}\left(N^{-2/3}\hat{k}^{-1/3}\right) \quad \text{for all } k \in [N].$$

[Bourgade - Erdős - Yau, Bourgade Mody - Pain]

Thm [Dumitriu - Edelman]

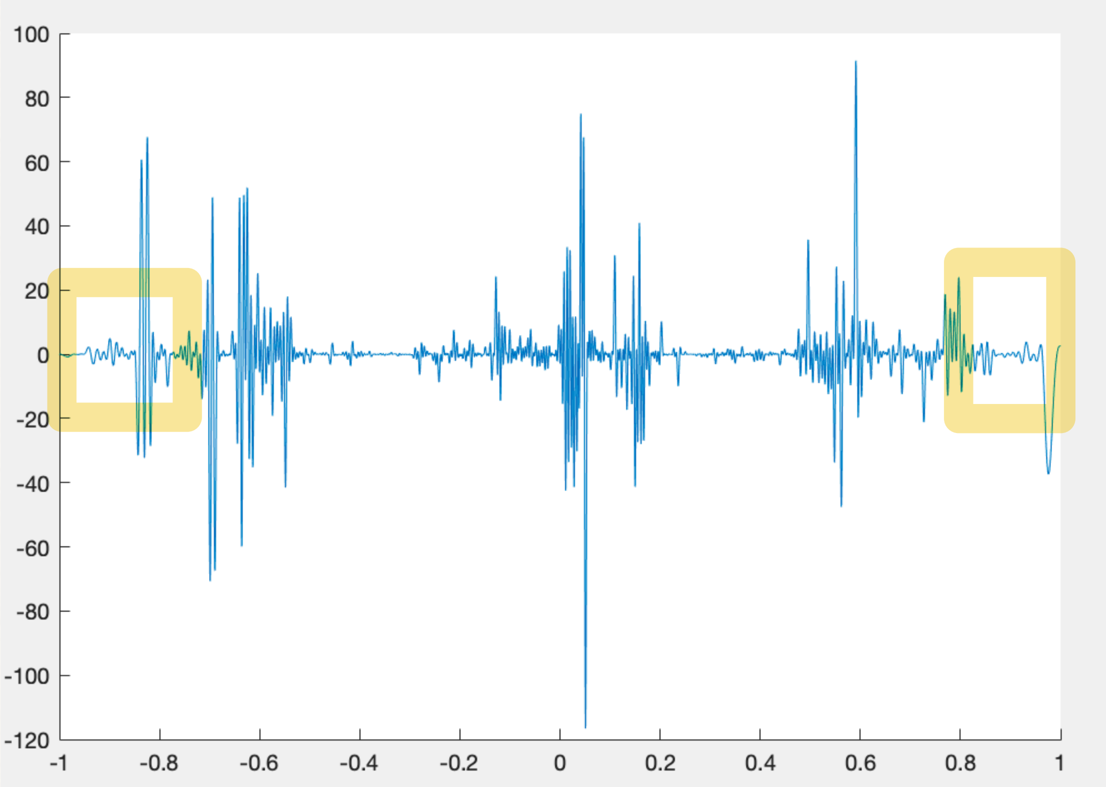
$(\lambda_j)_{j=1}^{N}$ are the eigenvalues of J_N where $J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$

$$\begin{cases} b_k = \dfrac{B_k}{\sqrt{2N\beta}} \\[2ex] a_k = \dfrac{\chi_{\beta k}}{\sqrt{4N\beta}} \simeq \sqrt{\dfrac{k}{4N}} + \dfrac{A_k}{\sqrt{2N\beta}} \end{cases}$$

$$\boxed{B_k, A_k \sim \mathcal{N}_{0,1} \quad \text{i.i.d.}}$$

$$\Longrightarrow \quad J = J_\infty + \frac{1}{\sqrt{2N\beta}} \begin{pmatrix} \ddots & & \\ & \text{iid} & \\ & & \ddots \end{pmatrix}.$$

Let $\Psi_N(x) := \left(\prod_{k=1}^{N-1}\sqrt{\frac{k}{4N}}\right)\det(x-J_N)\left(\frac{2N}{\pi}\right)^{1/4}e^{-Nx^2}$. [normalized GβE characteristic polynomial]



Sample of Ψ_N

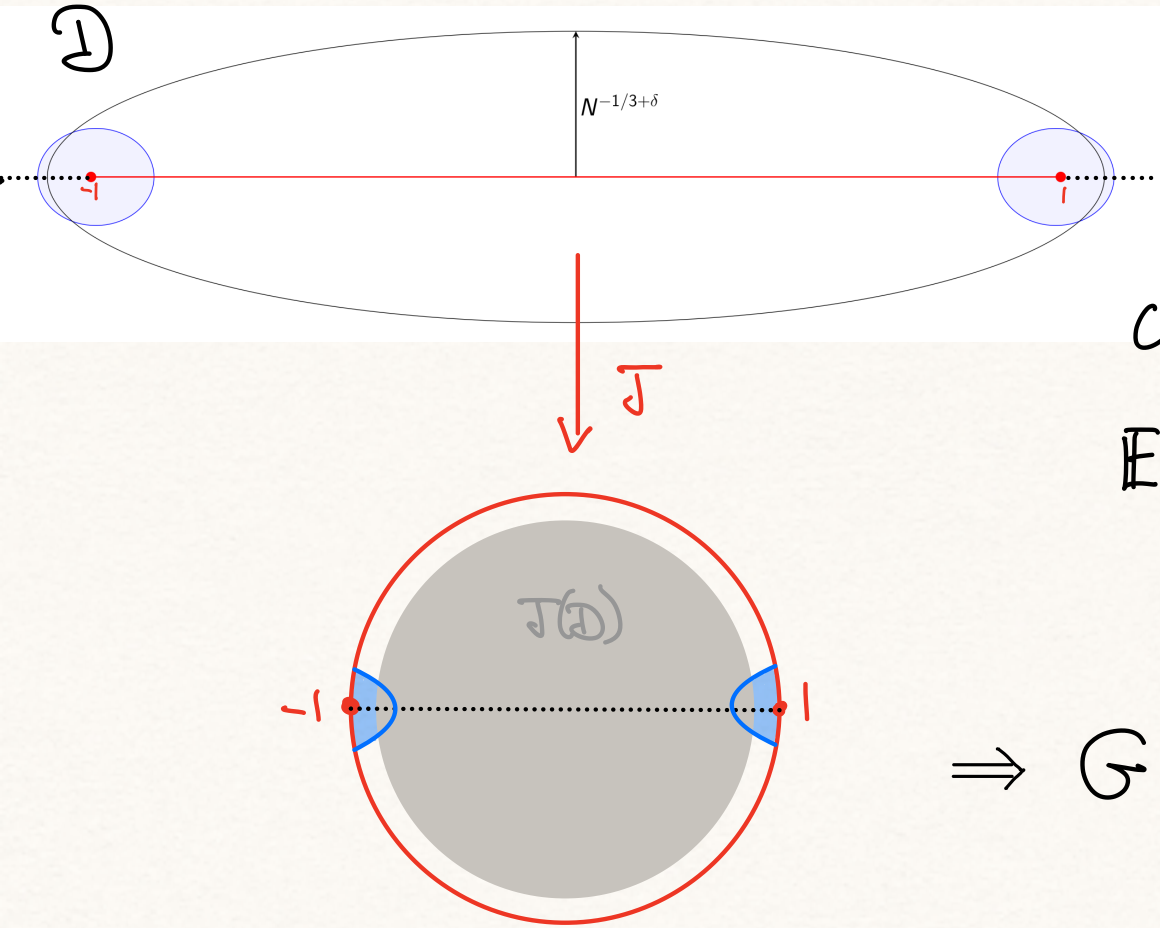
$N = 600$

$$\mathbb{E}\Psi_N(x) = \phi_N(x)$$

Hermite function

Thm 1 With overwhelming probability, uniformly for $x \in \mathcal{D}$

$$\Psi_N(x) = \phi_N(x)\,\frac{\exp\left(\beta^{-1/2}G(x)\right)}{\mathbb{E}\exp\left(\beta^{-1/2}G(x)\right)}\left(1+O(N^{-\varepsilon})\right).$$



$(G(z))_{z\in\mathbb{C}\setminus[-1,1]}$ is a centered <u>Gaussian</u> process with correlation kernel

$$\mathbb{E}\, G(x)G(z) = -\log\left(1 - J(x)J(z)\right)$$

$\Longrightarrow$ G is a log-correlated field on $[-1,1]$.

<u>Thm</u> [Edge] With overwhelming probability,

$$\psi_N\left(-1+\frac{\lambda}{2N^{2/3}}\right) = c\,N^{1/6}\,\frac{\exp\left(\sqrt{\frac{2}{\beta}}\,g_N\right)}{\mathbb{E}\exp\left(\sqrt{\frac{2}{\beta}}\,g_N\right)}\,SAi_0^{\beta}(\lambda)\left(1+\underset{N\to\infty}{o}(1)\right)$$

$$g_N \sim G(1)$$

$$\frac{g_N}{\sqrt{\frac{\log N}{3}}} \longrightarrow \mathcal{N}_{0,1}.$$

Define the Stochastic Airy operator $\mathcal{A}_\beta = -\partial_{tt} + t - \frac{2}{\sqrt{\beta}} \underbrace{dB_t}_{\text{white noise}}$ on $\mathbb{R}_+$

Thm [Rider - Ramirez - Virág]: $\mathcal{A}_\beta$ with Dirichlet b.c. has pure point spectrum $\underbrace{\Lambda_1 < \Lambda_2 < \Lambda_3 < \cdots}_{\text{Airy}_\beta \text{ point process}}$ etc

$\Lambda_1 \equiv$ Tracy - Widom β-distribution

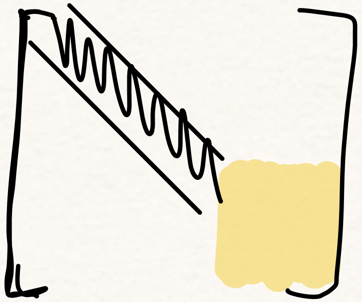
Prop: $\mathcal{A}_\beta SAi_t^\beta(\lambda) = -\lambda SAi_t^\beta(\lambda)$ and $\begin{cases} t \to SAi_t^\beta(\lambda) \in L^2(\mathbb{R}_+) \\ \mathbb{E} SAi_t^\beta(\lambda) = Ai(t-\lambda). \end{cases}$

$$(-\Lambda_j)_{j=1}^\infty = \text{zeros}\left(\lambda \to SAi_0^\beta(\lambda)\right)$$

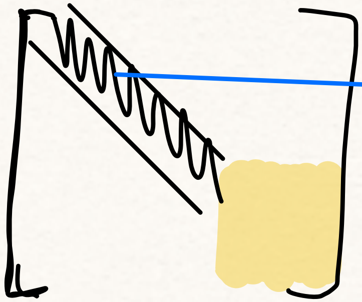
Corr: $\left((\lambda_j + 1) \cdot 2N^{2/3}\right)_{j=1}^N \longrightarrow (\Lambda_j)_{j=1}^\infty$ as $N \to \infty$ almost surely.

__Idea__ : $J = J_\infty + \frac{1}{\sqrt{2N\beta}} \begin{pmatrix} \ddots & & \\ & \text{iid} & \\ & & \ddots \end{pmatrix}$

In the transition window $x = 1 + \frac{\lambda N^{-2/3}}{2}$ $(\lambda \in \mathbb{R})$, $k = N - tN^{1/3}$ $(t \in \mathbb{R}_+)$

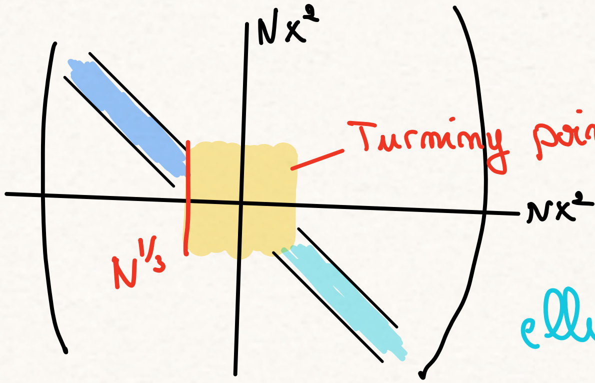
$J_\infty - xI =$  $\simeq \partial_{tt} - t - \lambda$

$N^{1/3}$

$J - xI =$  $\to e^{\sqrt{\frac{2}{\beta}} g_N - \frac{2}{\beta} \mathbb{E} g_N^2}$ (hyperbolic regime)

$\simeq \partial_{tt} - t - \lambda + \frac{2}{\sqrt{\beta}} dB_t$

For $x \in (-1, 1)$ hyperbolic regime



elliptic regime

Bulk asymptotics

Recall that $\phi_N\left(x+\frac{\lambda}{N\rho(x)}\right) \simeq \dfrac{\cos\left(\pi N F(x) - \frac{\theta(x)}{2} + \pi\lambda\right)}{\sqrt{\pi/2}\,(1-x^2)^{1/4}}$ for $x \in (-1,1)$.

<u>Thm</u> With overwhelming probability, for $x \in \left[-1+\frac{C_N}{N^{2/3}}, 1-\frac{C_N}{N^{2/3}}\right]$, uniformly for $x \in K$

$$\Psi_N\left(x+\frac{\lambda}{N\rho(x)}\right) \simeq \frac{1}{\sqrt{\frac{\pi}{2}}(1-x^2)^{1/4}} \operatorname{Re}\left\{\exp\left(\pi N F(x) - \frac{\theta(x)}{2} + \frac{M_N(x)}{\sqrt{\beta}} - \frac{[M_N(x)]}{2\beta} + \pi \Omega^{\beta}(\lambda; x) + \underset{N\to\infty}{o(1)}\right)\right\}$$

M_N is a complex-valued log-correlated field:

- $$[M_N(x), \overline{M_N(z)}] = \begin{cases} -\log\left(1 - J(x)\overline{J(z)}\right) + \underset{N\to\infty}{o(1)} & \text{if } |z-x| \gg N^{-2/3} \\ -\log|x-z|^{-1} + \underset{N\to\infty}{\mathcal{O}(1)} & \text{else} \end{cases}$$

$$[M_N(x), M_N(z)] \simeq \log\left(1 - J(x)J(z)\right).$$

- Consider the equation

$$\begin{cases} d\Omega_t^\beta(\lambda) = i\,\frac{\lambda\, dt}{2\sqrt{t}} + \sqrt{\frac{2}{\beta t}}\left\{\left(1-e^{-i\Omega_t^\beta(\lambda)}\right)\underbrace{dW_t}_{\text{complex Brownian motion}}\right\} \quad ; \; t\in[0,1] \qquad \lambda\in\mathbb{R} \\ \Omega_0^\beta(\lambda) = 0 \end{cases}$$

$$\Omega^\beta(\lambda; x) = \Omega_1^\beta(\lambda) \text{ driven by } \left(W_t(x)\right)_{t\in[0,1]}.$$

- For $x \in \left[-1+\frac{c_N}{N^{2/3}}, 1-\frac{c_N}{N^{2/3}}\right]$,

$$\left(\frac{\operatorname{Im} M_N(x)}{\sqrt{\log N\,(1-x^2)^{3/2}}}, \frac{\operatorname{Re} M_N(x)}{\sqrt{\log N\,(1-x^2)^{1/2}}}\right) \xrightarrow[N\to\infty]{} \mathcal{N}\left(0, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) \perp\!\!\!\perp \Omega(\cdot\,; x)$$

- The Sine_β process can be defined as follows; let Θ be a uniform random variable in $[0,1]$ independent of $(\Omega_t^\beta)_{t\in[0,1]}$,

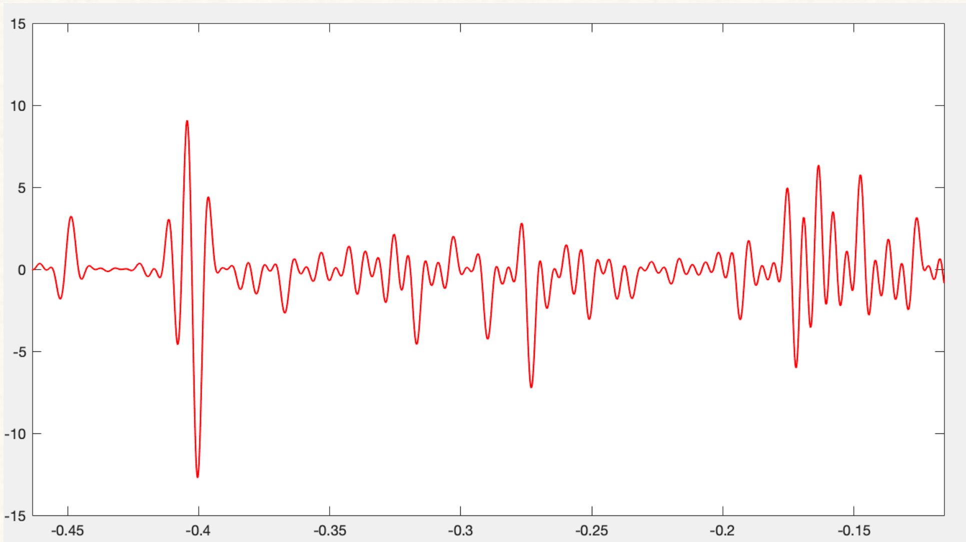
$$\mathrm{Sine}_\beta = \left\{\lambda\in\mathbb{R} : \Omega_1^\beta(\lambda) = \Theta\right\}.$$

Thm: Fix $x \in (-1,1)$. Almost surely as $N \to \infty$,

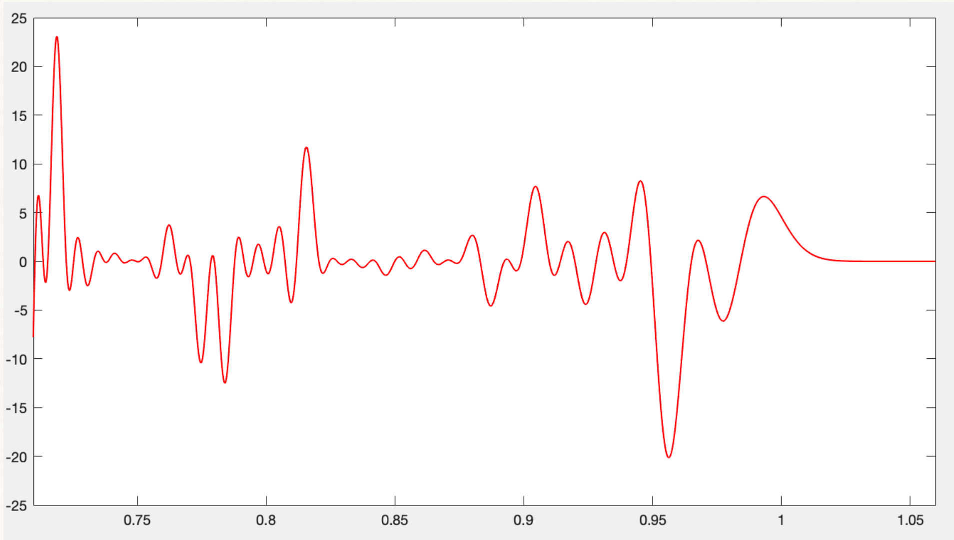
$$\left\{ (\lambda_j - x) \cdot N \rho(x) \right\}_{j=1}^{N} \longrightarrow \text{Sine}_\beta \text{ point process}$$

$$\frac{\psi_N\left(x + \frac{\lambda}{N\rho(x)}\right)}{\psi_N(x)} \longrightarrow \prod_{x_j \in \text{Sine}_\beta}^{*} \left(1 - \frac{\lambda}{x_j}\right) = \text{Stochastic } \xi_\beta\text{-function}$$

[Valkó - Virág]



Stochastic ξ_β - function.



Stochastic Airy function $\lambda \to SAi^\beta(\lambda)$.

$$M_N(x) := G_N(x) - W_N(x) + \sigma(x)$$

$$J = J_\infty + \frac{1}{\sqrt{2N\beta}} \begin{pmatrix} \ddots & \ddots & \ddots \\ A_k & B_k & A_k \\ \ddots & \ddots & \ddots \end{pmatrix} \qquad Z_k := \frac{B_k + \lambda_k^+(x) A_k}{\sqrt{2}}$$

$$G_N(x) \simeq \sum_{k=1}^{N} \frac{Z_k(x)}{\sqrt{Nx^2 - k}} \qquad W_N(x) \simeq \sum_{k=Nx^2}^{N} \frac{\overline{Z_k(x)}}{\sqrt{Nx^2 - k}} e^{-2i \operatorname{Im} \Psi_k(x)}$$

Idea: Define $\begin{pmatrix} \xi_k \\ \overline{\xi_k} \end{pmatrix} := \begin{pmatrix} \lambda_+ & \lambda_- \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \Psi_{k+1} \\ \Psi_k \end{pmatrix}$ $\quad \begin{cases} \Psi_k = 2 \operatorname{Re} \xi_k \\ \lambda_\pm = e^{\pm i \theta_k(x)} \end{cases}$

$$\xi_{k+1} \simeq \left(1 - \delta_k^2 + \frac{\delta_k}{\sqrt{\beta}} Z_k\right)\xi_k + \left(\delta_k^2 + \frac{\delta_k}{\sqrt{\beta}} \overline{Z_k}\, e^{-2i\theta_k}\right) e^{-2i\chi_k} \overline{\xi_k} \qquad \delta_k = \frac{1}{\sqrt{k - Nx^2}}$$

$$\boxed{\xi_k = \exp H_k}$$

3 Open questions

- Show that $$\frac{|\Phi_N(x)|^{\gamma}}{\mathbb{E}|\Phi(x)|^{\gamma}}\,dx \longrightarrow \mathrm{GMC}_{\gamma/\sqrt{\beta}} \text{ on } [-1,1]$$ for $\gamma < \sqrt{2\beta}$.

- What is the spectral of the Jacobi operator

$$J = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \qquad \begin{array}{l} b_k \sim \mathcal{N}_{0,2} \\ a_k \sim \chi_{\beta k} \end{array} \quad \text{i.i.d.}$$

- Show that

$$\max_{x\in\mathbb{R}} \sqrt{\beta}\log|\Phi_N(x)| = \log N - \frac{3}{4}\log\log N + \mathrm{Gumbel} + \mathcal{Z} + c_\beta + \underset{N\to\infty}{o(1)}$$

Thank you!