Harmonically confined Riesz gas in one dimension

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Collaborators and References

- S. Agarwal, J. Kethepalli, A. Dhar, M. Kulkarni, A. Kundu (ICTS, Bangalore, India)
- D. Mukamel (Weizmann Institute, Israel)
- G. Schehr (LPTHE, Sorbonne Univ., France)
- A. Flack (LPTMS, Univ. Paris Saclay, France) $\rightarrow$ related work

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Refs:

- S. Agarwal, A. Dhar, M. Kulkarni, S.M., D. Mukamel, G. Schehr, "Harmonically confined particles with long-range repulsive interactions", Phys. Rev. Lett. 123, 100603 (2019)
- J. Kethepalli, M. Kulkarni, A. Kundu, S.M., D. Mukamel, G. Schehr, "Harmonically confined long-ranged interacting gas in the presence of a hard wall", J. Stat. Mech. 103209 (2021)
- J. Kethepalli, M. Kulkarni, A. Kundu, S.M., D. Mukamel, G. Schehr, "Edge fluctuations and third-order phase transition in harmonically confined long-ranged systems", J. Stat. Mech. 033203 (2022)

Plan

I. Brief introduction to the harmonically confined Riesz gas in 1-d

II. Average density in the large N limit

Agarwal et. al., Phys. Rev. Lett. 123, 100603 (2019)

III. Average density in the large N limit in the presence of a hard wall

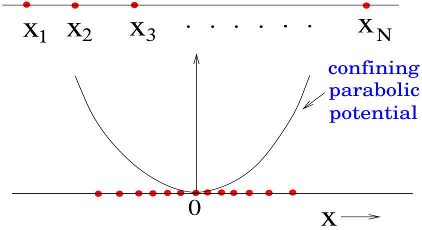
Kethepalli et. al., J. Stat. Mech. 103209 (2021)

IV. Statistics of the position of the rightmost particle

$x_{\rm max} = \max[x_1, x_2, \ldots, x_N] \Longrightarrow$ **explicit** large deviation tails

Kethepalli et. al., J. Stat. Mech. 033203 (2022)

Harmonically confined Riesz gas in $d = 1$



N particles on a line with pairwise repulsive interaction (M. Riesz, 1948)

$$E[\{x_i\}] = \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{J\,\mathrm{sgn}(k)}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|^k} \quad , \quad k > -2$$

For a recent survey: M. Lewin, JMP, 63, 061101 (2022)

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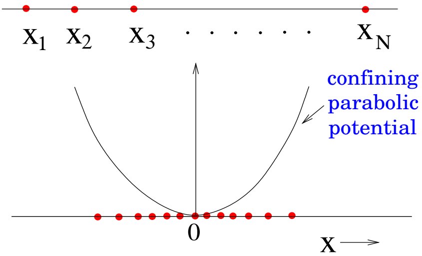

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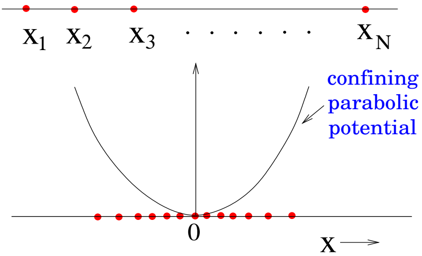

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Stationary Gibb's measure: $P[\{x_i\}] = \frac{1}{Z_N}\, e^{-\beta\, E[\{x_i\}]}$

where $Z_N = \int dx_1\, dx_2 \ldots dx_N\, e^{-\beta\, E[\{x_i\}]} \longrightarrow$ partition function

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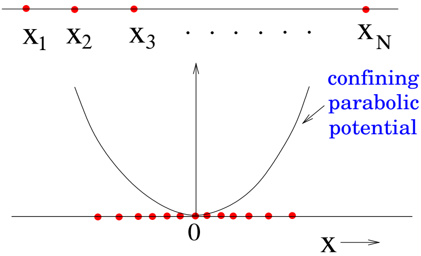

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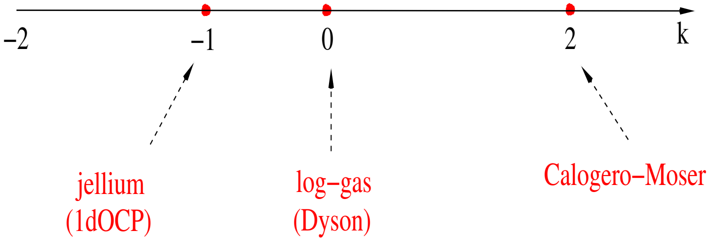
Applications: cold atoms, random matrix theory, integrable systems, dipolar Bose gas, ionic systems, ...

Well known models for special values of $k > -2$



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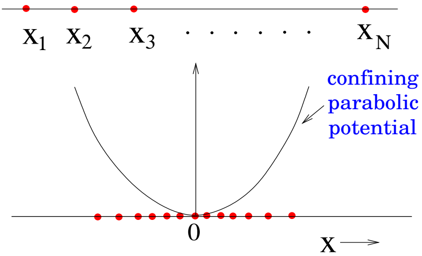
Observables of interest



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Two main observables:

- Average density: $\langle \rho_N(x)\rangle = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(x - x_i)\right\rangle$ (normalized to unity)
- Statistics of $x_{\max} = \max[x_1, x_2, \ldots, x_N]$

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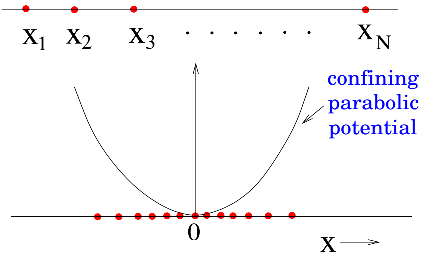

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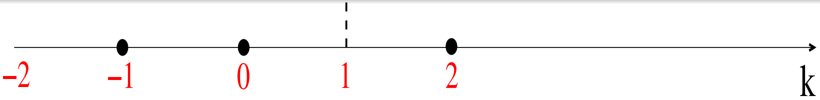
We will be mostly interested in the large N limit

Average Density

Known results for special k values



Average density (finite support): $\langle \rho_N(x) \rangle \to \frac{1}{L_N}\, \rho_k^*\left(\frac{x}{L_N}\right)$

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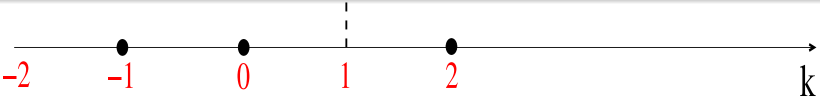

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  $\rho_{-1}^*(y) = \frac{1}{2}$ for $-1 \leq y \leq 1 \Longrightarrow$ flat density

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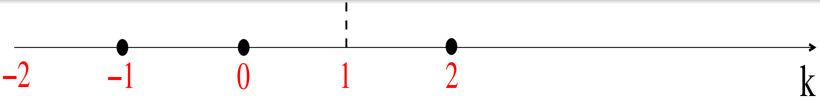

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- $k \to 0^+$ (Log-gas): $L_N \sim \sqrt{N}$ (set $J = 1/k$)

  $\rho_0^*(y) = \frac{1}{\pi}\sqrt{2 - y^2}$ for $-\sqrt{2} \le y \le \sqrt{2} \Longrightarrow$ Wigner semi-circle

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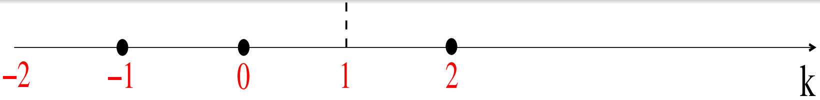

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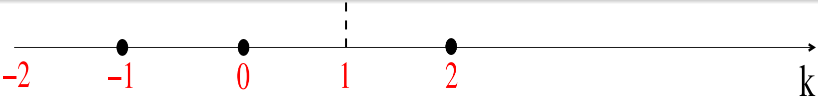

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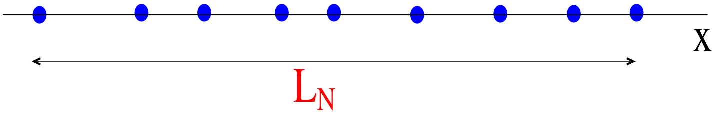
- $k > 1$: $L_N \sim N^{k/(k+2)}$ (set $J = 1$)

  $\rho_{k>1}^*(y) = [2\,\zeta(k)\,(k+1)]^{-1/k}\, \left(l_k^2 - y^2\right)^{1/k}$ for $-l_k \le y \le l_k$

Hardin, Leblé, Saff & Serfaty, Constr. Approx. 48, 61 (2018)

Why $k = 1$ is special? Scaling argument



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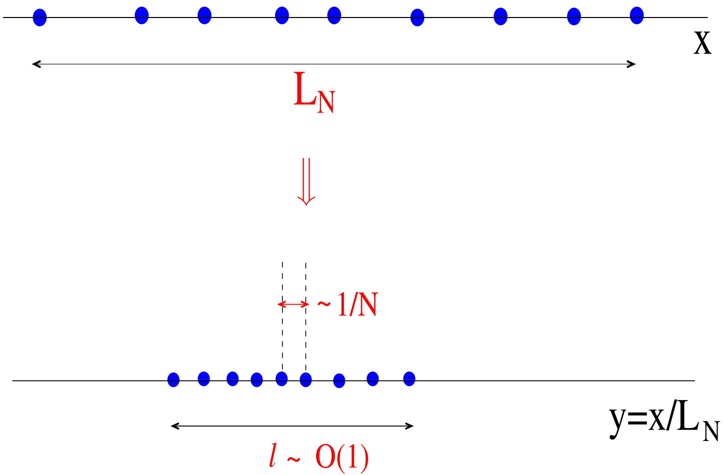
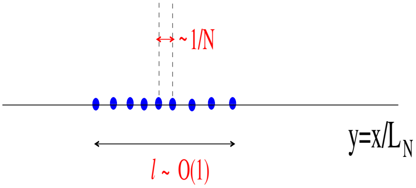


The global scale L_N depends on k

Scaling argument to determine L_N



Energy function:

$$E[\{x_i\}] = \underbrace{\frac{1}{2}\sum_{i=1}^{N} x_i^2}_{T_1} + \underbrace{\frac{J\,\mathrm{sgn}(k)}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|^k}}_{T_2}$$

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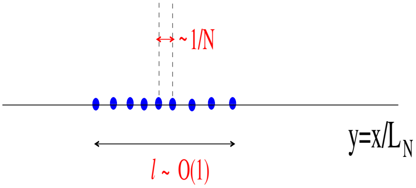

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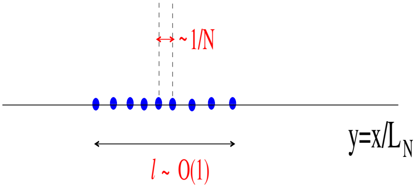

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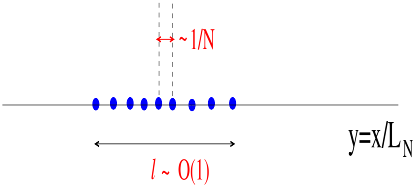

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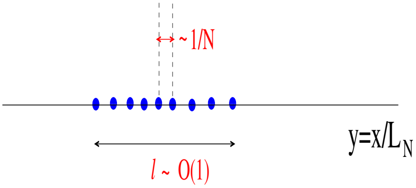

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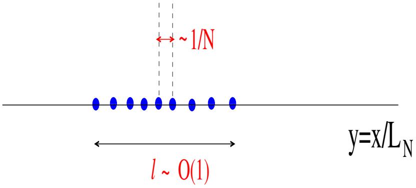

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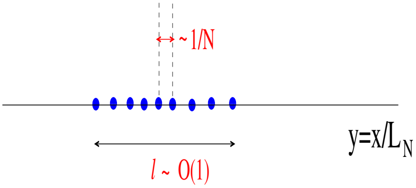

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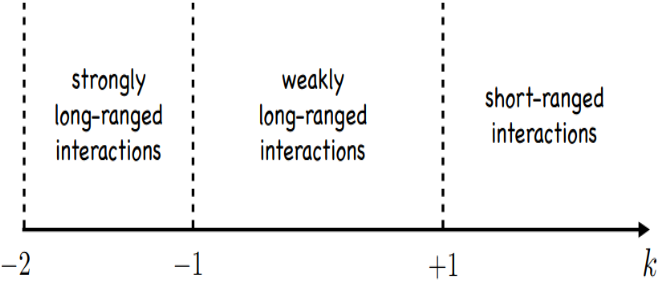
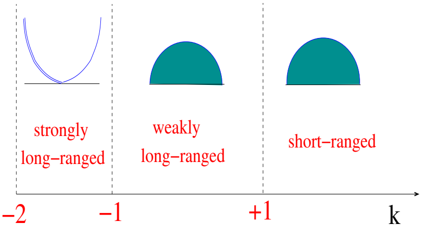
Matching the two terms $T_1 \sim T_2$ determines the appropriate scale L_N

k-dependence of the global scale L_N



$$L_N \approx \begin{cases} N^{\alpha_k} & \text{for} \quad k \neq 1 \\ (N \ln N)^{1/3} & \text{for} \quad k = 1 \end{cases} \qquad \alpha_k = \begin{cases} \frac{1}{k+2} & \text{for} \quad -2 < k < 1 \\ \frac{k}{k+2} & \text{for} \quad k > 1 \end{cases}$$

Our main result: Exact av. density for all $k > -2$



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The scaling function: $\boxed{\rho_k^*(y) = A_k\, \left(l_k^2 - y^2\right)^{\gamma_k}}$, $\quad -l_k \le y \le l_k$

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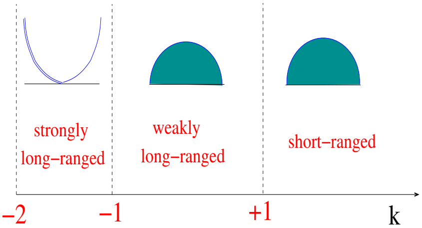

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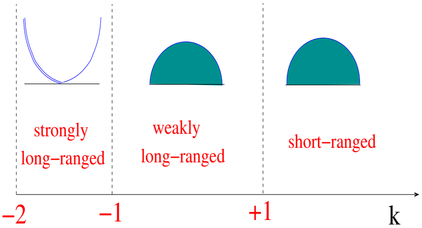

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$$\gamma_k = \begin{cases} \frac{k+1}{2} & \text{for} \quad -2 < k < 1 \\ \frac{1}{k} & \text{for} \quad k > 1 \end{cases}$$

Agarwal et. al., PRL, 123, 100603 (2019)

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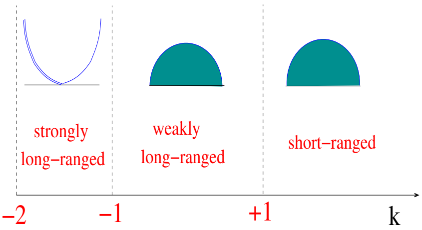

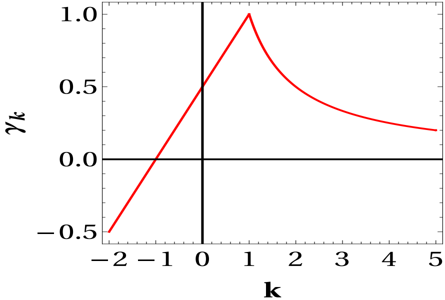

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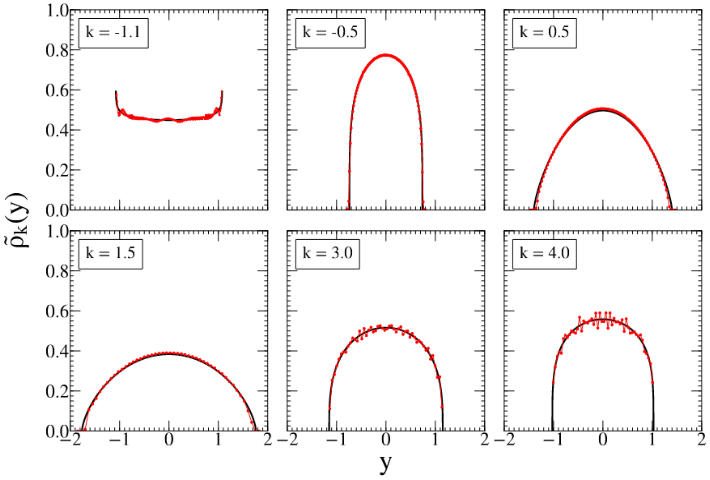
Numerical simulations



Brief outline of the derivation

The partition function in the Riesz gas:

$$Z_N = \int dx_1\, dx_2\, \ldots dx_N\, e^{-\beta\, E[\{x_i\}]}$$

$$\boxed{E[\{x_i\}] = \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{J\,\mathrm{sgn}(k)}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|^k}} \quad , \quad k > -2$$

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The partition function in the Riesz gas:

$$Z_N = \int dx_1\, dx_2\, \ldots dx_N\, e^{-\beta\, E[\{x_i\}]}$$

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For large $N \Longrightarrow$ continuum/hydrodynamic approximation

Two steps:

- Fix a macroscopic density profile $\rho(x)$ (normalised to unity) and sum over all microscopic configs. of x_i's corresponding to $\rho(x)$
- Integrate (functionally) over all possible (normalised to unity) macroscopic density profiles

Coarse-grained description

The partition function (after re-scaling $x \to y\, L_N$ where $y \sim O(1)$):

$$Z_N \approx \int \mathcal{D}\rho(y)\, \delta\left(\int dy\, \rho(y) - 1\right)\, e^{-\beta\, N^{2\alpha_k+1}\, \mathcal{E}_k[\rho(y)] - N \int dy\; \rho(y)\ln(\rho(y))}$$

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where $\alpha_k = k/(k+2)$ for $k > 1$ and $\alpha_k = 1/(k+2)$ for $-2 < k < 1$

The effective energy (setting $J = 1$):

$$\mathcal{E}_k\left[\rho(y)\right] \approx \frac{1}{2}\int dy\; y^2\, \rho(y) + \begin{cases} \zeta(k) \int dy\; [\rho(y)]^{k+1}, & \text{for} \quad k > 1 \\ \int dy\; [\rho(y)]^2, & \text{for} \quad k = 1 \\ \frac{\text{sgn}(k)}{2} \int dy' dy \frac{\rho(y')\rho(y)}{|y-y'|^k} & \text{for} -2 < k < 1. \end{cases}$$

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For $k \geq 1$ the interaction energy is local (short-ranged), while for $k < 1$ the interaction energy is non-local (long-ranged)

Saddle point computation

Neglecting the entropy term for large N and replacing the delta function by its integral representation:

$$Z_N \sim \int d\mu \int \mathcal{D}\rho(y) \exp\left[-\beta\, N^{2\alpha_k+1}\, S_k\left(\rho(y)\right) + o\left(N^{2\alpha_k+1}\right)\right]$$

where the action $S_k\left(\rho(y)\right) = \mathcal{E}_k\left[\rho(y)\right] - \mu\, \left(\int dy\, \rho(y) - 1\right)$

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Minimize the effective action: $\boxed{\dfrac{\delta S_k[\rho(y)]}{\delta\rho(y)} = 0} \Rightarrow \rho_k^*(y)$

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$$\Longrightarrow \text{Av. density} \lim_{N\to\infty}\langle\rho_N(y)\rangle \to \rho_k^*(y)$$

Explicit minimizer $\rho_k^*(y)$

- $k \geq 1$: Saddle point equation: $\frac{y^2}{2} + \zeta(k)\,(k+1)\,[\rho_k^*(y)]^k = \mu$

$\Longrightarrow \rho_k^*(y) = (2\,\zeta(k)\,(k+1))^{-1/k}\,(2\mu - y^2)^{1/k}$ with $-\sqrt{2\mu} \leq y \leq \sqrt{2\mu}$

The Lagrange multiplier μ is fixed from $\int \rho_k^*(y) dy = 1$

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Assuming the support $[-l_k, l_k]$ and taking one derivative

$$\Longrightarrow \qquad \mathrm{PV}\,\int_{-l_k}^{l_k} \frac{\mathrm{sgn}(y'-y)}{|y-y'|^{k+1}}\,\rho_k^*(y')\,dy' = -\frac{y}{|k|}$$

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Explicit solution of the singular integral equation using Sonin formula

Sonin inversion formula

Singular integral equation: $$\boxed{\mathrm{PV}\int_0^L \frac{\mathrm{sgn}(z'-z)}{|z-z'|^p}\, f(z')\, dz' = h(z)}$$

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General solution: $$f(z) = c_0\,\left[z\,(L-z)\right]^{p/2-1} + u(z)$$

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General solution: $$f(z) = c_0\, [z\,(L-z)]^{p/2-1} + u(z)$$

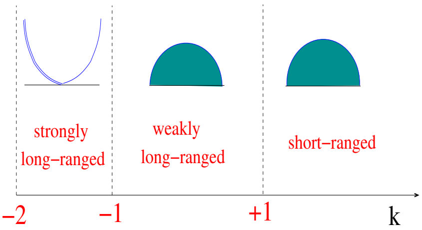
where c_0 is an arbirary constant and the inhomogeneous solution

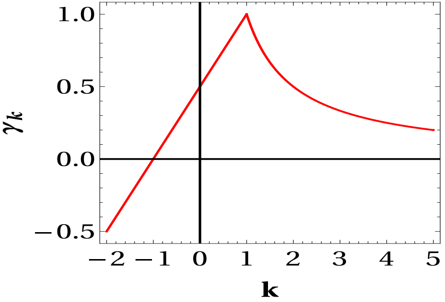
$$u(z) = A(p)\, z^{p/2-1}\, \frac{d}{dz} \int_0^L dt\, t^{1-p} (t-z)^{p/2}\, \frac{d}{dt} \int_0^t y^{p/2} (t-y)^{p/2-1}\, h(y)\, dy$$

with $A(p) = \frac{2\,\sin(\pi\, p/2)}{\pi\, p\, \mathrm{B}(p/2, p/2)}$

N. Ya Sonin (1954)

Our main result: Exact av. density for all $k > -2$





The scaling function: $\boxed{\rho_k^*(y) = A_k\ \left(l_k^2 - y^2\right)^{\gamma_k}}$, $\quad -l_k \le y \le l_k$

A_k and l_k are explicit and

$$\gamma_k = \begin{cases} \frac{k+1}{2} & \text{for} \quad -2 < k < 1 \\ \\ \frac{1}{k} & \text{for} \quad k > 1 \end{cases}$$

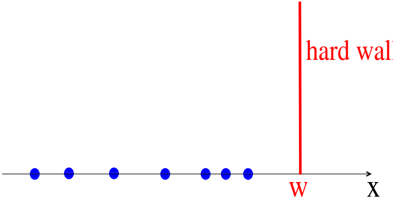
Agarwal et. al., PRL, 123, 100603 (2019)

Average Density

in the presence of a

hard wall

Riesz gas in the presence of a hard wall

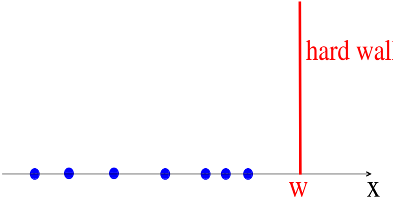


$x_i \leq W \quad \forall i = 1, 2, \ldots, N$

Energy function:

$$E[\{x_i\}] = \frac{1}{2}\sum_{i=1}^{N} x_i^2 + \frac{J\,\mathrm{sgn}(k)}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|^k}$$

Riesz gas in the presence of a hard wall



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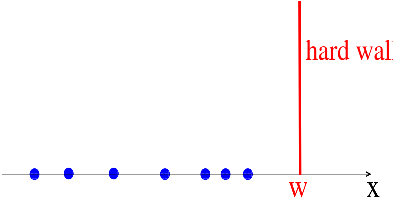
Partition function:

$$\boxed{Z_N(W) = \int_{-\infty}^{W}\ldots\int_{-\infty}^{W} dx_1\, dx_2\, \ldots dx_N\, e^{-\beta\, E[\{x_i\}]}}$$

Riesz gas in the presence of a hard wall



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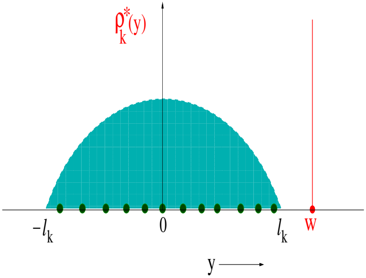
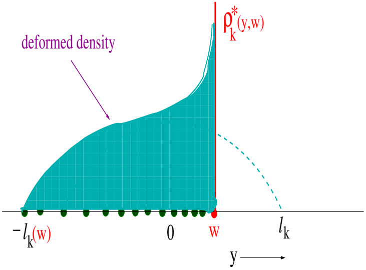
Cumulative prob. dist. of $x_{\max}$:

$$\mathrm{Prob.}\,[x_{\max} \leq W] = \frac{Z_N(W)}{Z_N(\infty)}$$

Deformed saddle point density $\rho_k^*(y, w)$





- For $w \geq l_k$, the av. density remains as in the unconstrained case:

$$\boxed{\rho_k^*(y) = A_k \left(l_k^2 - y^2\right)^{\gamma_k}} \quad , \quad -l_k \leq y \leq l_k$$

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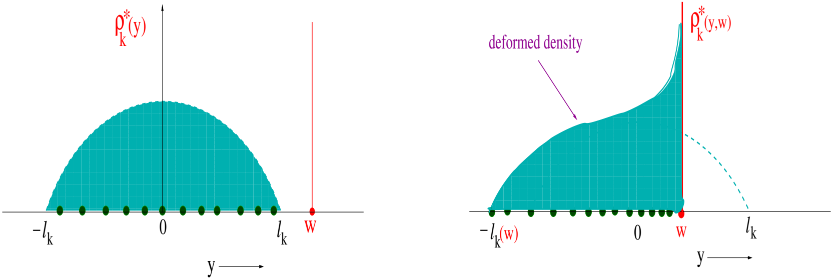

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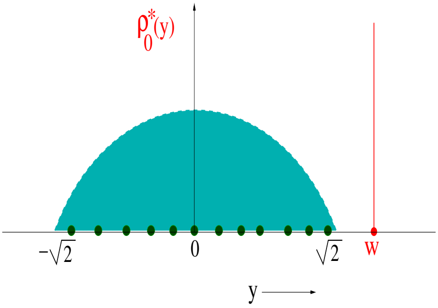
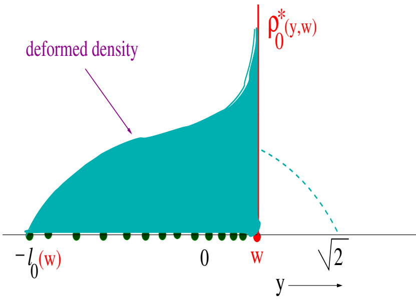
- For $w < l_k$, the av. density gets deformed to

$$\boxed{\rho_k^*(y, w) = ?} \quad -l_k(w) \leq y \leq w$$

Known results: Log-gas $k \to 0^+$



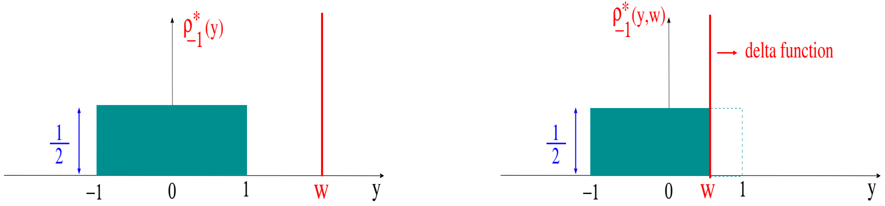


$$\rho_0^*(y,w) = \begin{cases} \frac{1}{\pi}\sqrt{2-y^2} & \text{with} \quad -\sqrt{2} \le y \le \sqrt{2} \quad \text{for } w > \sqrt{2} \\ \frac{1}{2\pi}\sqrt{\frac{y+l_0(w)}{w-y}}\,[w+l_0(w)-2y] & \text{with} \quad -l_0(w) \le y \le w \quad \text{for } w < \sqrt{2} \end{cases}$$

where $l_0(w) = \frac{2\sqrt{w^2+6}-w}{2}$

D.S. Dean & S. M., PRL, 97, 160201 (2006)

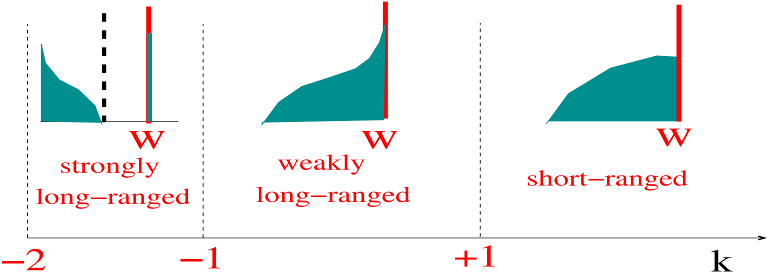
Known results: jellium model $k = -1$



$$\rho^*_{-1}(y,w) = \begin{cases} \frac{1}{2} & \text{with} \quad -1 \leq y \leq 1 \quad \text{for} \quad w > 1 \\ \frac{1}{2} + \frac{(1-w)}{2}\,\delta(w-y) & \text{with} \quad -1 \leq y \leq w \quad \text{for} \quad -1 < w < 1 \\ \delta(w-y) & \text{with} \quad y \leq w \quad \text{for} \quad w < -1 \end{cases}$$

A. Dhar, A. Kumdu, S.M., S. Sabhapandit, G. Schehr, PRL, 119, 060601 (2017)

Our result: explicit density for all $k > -2$



short-ranged: $k > 1$

$$\rho_k^*(y,w) = \left[2(k+1)\zeta(k)\right]^{-1/k} \left(l_k^2(w) - y^2\right)^{1/k} \quad , \qquad -l_k(w) \le y \le w$$

$l_k(w)$ is fixed from the normalization: $\int_{-l_k(w)}^{w} \rho_k^*(y,w)\, dy = 1$

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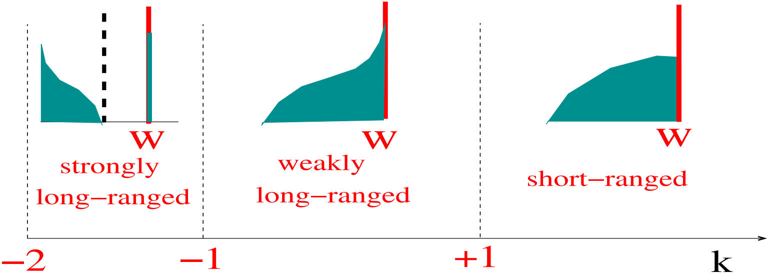
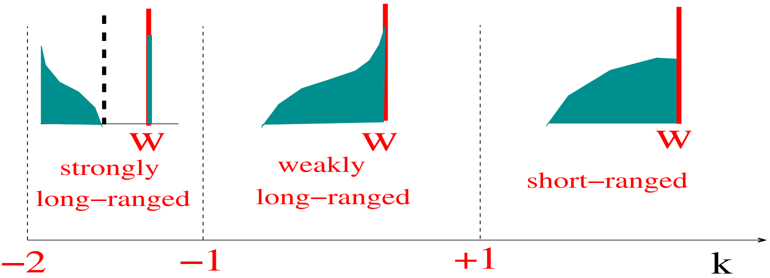


weakly long-ranged: $-1 < k < 1$

$$\rho_k^*(y, w) = A_k \frac{(a_k(w)-y)(l_k(w)+y)^{(k+1)/2}}{(w-y)^{(1-k)/2}} \quad , \quad -l_k(w) \le y \le w$$

A_k, $a_k(w)$ and $l_k(w) \Longrightarrow$ **explicit**

Our result: explicit density for all $k > -2$

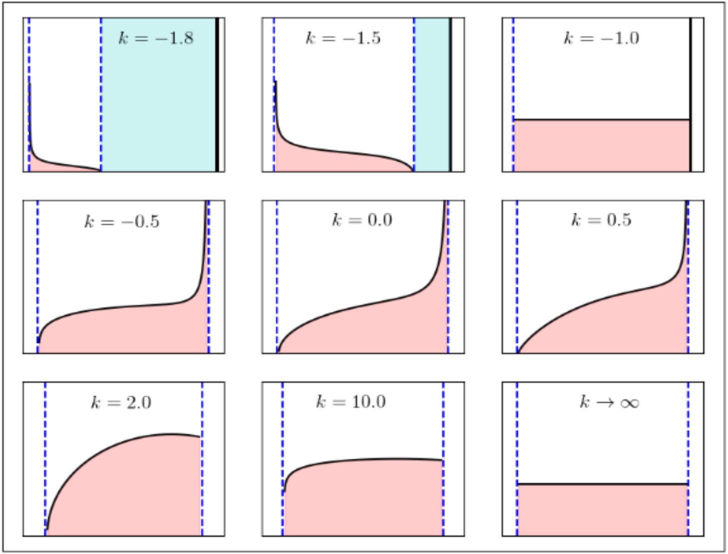


strongly long-ranged: $-2 < k < -1$

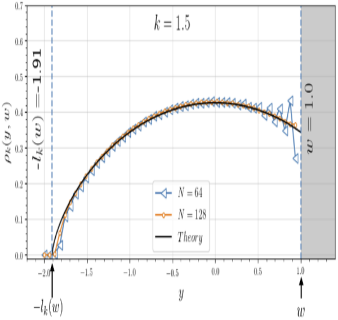
$$\rho_k^*(y,w) = \begin{cases} A_k \frac{(l_k(w)+y)^{\frac{k+1}{2}} (\bar{l}_k(w)-y)^{\frac{k+3}{2}}}{(w-y)} \mathbb{I}[-l_k(w) < y \le \bar{l}_k(w)] + D_k^*(w)\,\delta(w-y), & \text{for} \quad w > w_c(k) \\ \delta(w-y), & \text{for} \quad w < w_c(k) \end{cases}$$

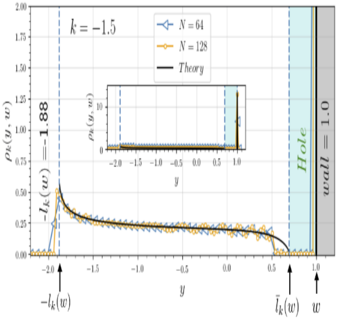
Kethepalli et. al., J. Stat. Mech. 103209 (2021)

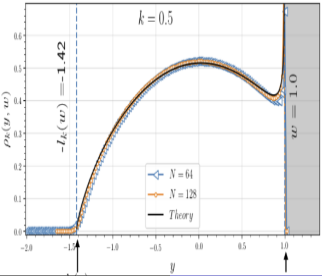
Exact av. density for different $k > -2$



Numerical simulations



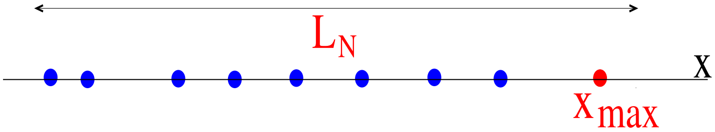




Distribution of $x_{\max}$

Explicit large deviation tails

Statistics of $x_{\max}$ in the Riesz gas



After rescaling: $y_{\max} = x_{\max}/L_N \sim O(1)$ where

$$L_N \approx \begin{cases} N^{\alpha_k} & \text{for} \quad k \neq 1 \\ (N \ln N)^{1/3} & \text{for} \quad k = 1 \end{cases} \qquad \alpha_k = \begin{cases} \frac{1}{k+2} & \text{for} \quad -2 < k < 1 \\ \frac{k}{k+2} & \text{for} \quad k > 1 \end{cases}$$

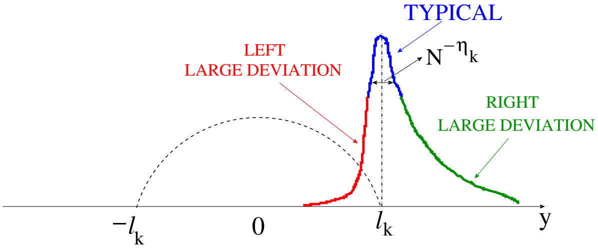
Probability density of $y_{\max}$:

$$\boxed{P_k(y_{\max} = w, N) = ?}$$ in the large N limit for general $k > -2$

Results known for $k \to 0$ (log-gas) and $k = -1$ (jellium)

For a review see S.M. & G. Schehr, J. Stat. Mech. 01012 (2014)

Our new result for arbitrary $k > -2$



$$P_k\left[y_{\max} = w, N\right] \approx \begin{cases} \exp\left[-\beta\, N^{2\,\alpha_k+1}\,\Phi_-(w,k)\right] & \text{for} \quad l_k - w \gtrsim O(1) \\ N^{\eta_k}\, f^{(k)}\left(N^{\eta_k}\,(w-1)\right) & \text{for} \quad |w - l_k| \lesssim O(N^{-\eta_k}) \\ \exp\left[-\beta\, N^{2\,\alpha_k}\,\Phi_+(w,k)\right] & \text{for} \quad w - l_k \gtrsim O(1). \end{cases}$$

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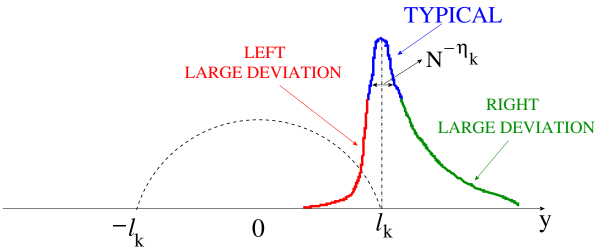

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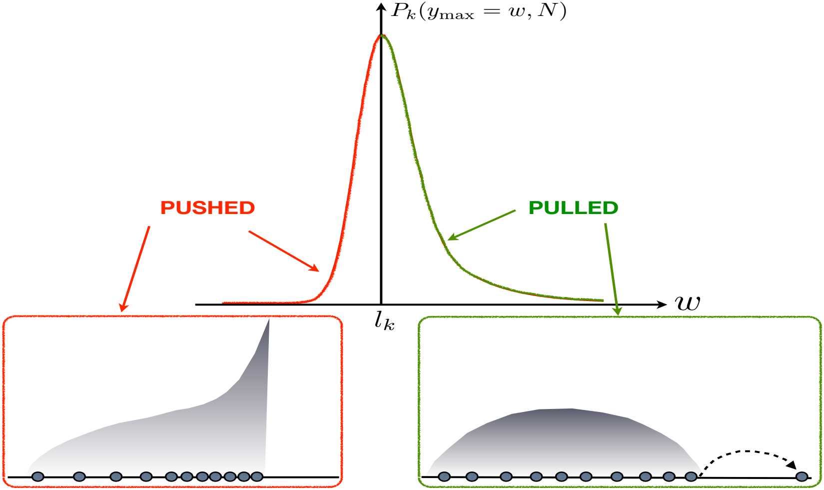
The rate functions $\Phi_{\pm}(w,k) \Longrightarrow$ fully **explicit** for all $k > -2$

$\eta_{\mathbf{k}}$ and $f^{(k)}(z) \Longrightarrow$ numerical except for $k \to 0^+$ and $k = -1$

Kethepalli et. al. J. Stat. Mech. 033203 (2022)

Transition between Pushed and Pulled phases



PUSHED gas **PULLED gas**

Large deviation functions for general $k > -2$

Left large deviation function as an effective free energy:

$$\lim_{N\to\infty} -\frac{1}{N^{2\alpha_k+1}} \log\left(\mathrm{Prob.}[y_{\max} \le w, N]\right) = \begin{cases} \Phi_-(w,k), & w < l_k \\ 0 & w > l_k, \end{cases}$$

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$$\zeta_k = \begin{cases} 3 & \text{for} \quad -2 < k < 1 \quad \text{(long-ranged)} \\ 2 + \frac{1}{k} & \text{for} \quad k > 1 \quad \text{(short-ranged)} \end{cases}$$

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Since $\lceil 2 + \frac{1}{k} \rceil = 3$, the transition is **3**-rd order for all $k > -2$

$\Longrightarrow$ **Universal** for all $k > -2$

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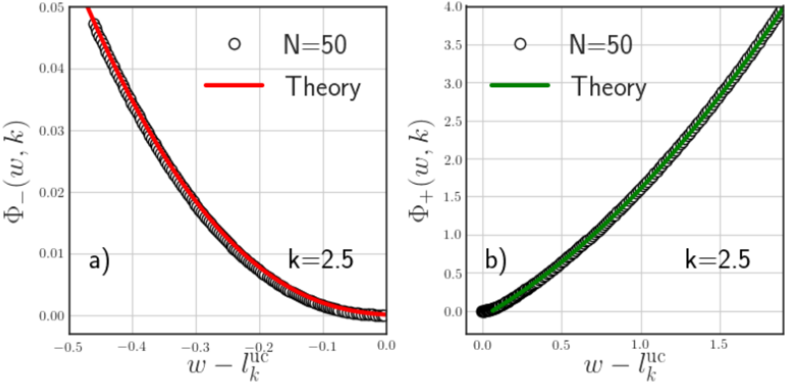
Numerical Confirmations



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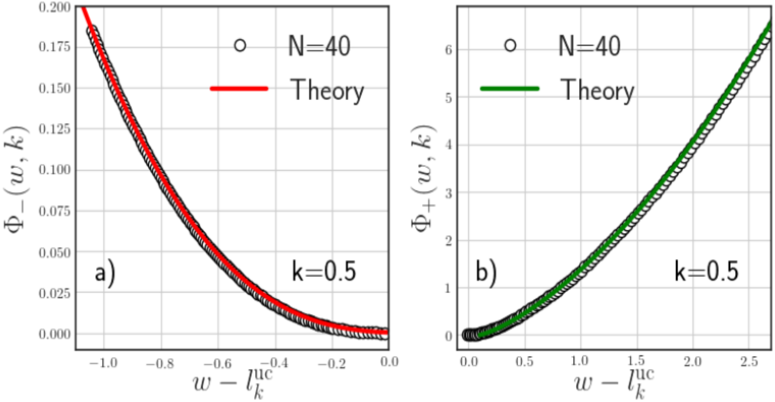

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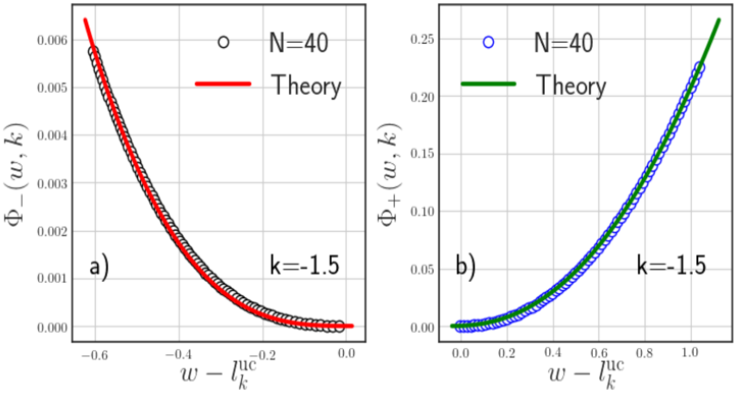
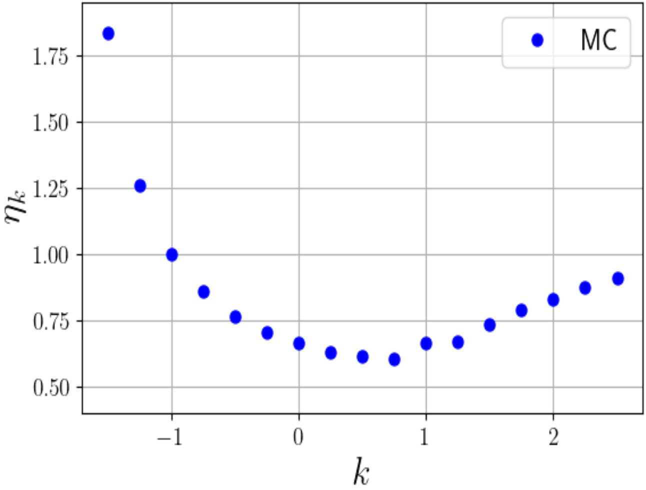


Monte Carlo estimate of the exponent η_k



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Summary and Conclusions

- Exact average large-N density for a harmonically confined Riesz gas on a line for all $k > -2$

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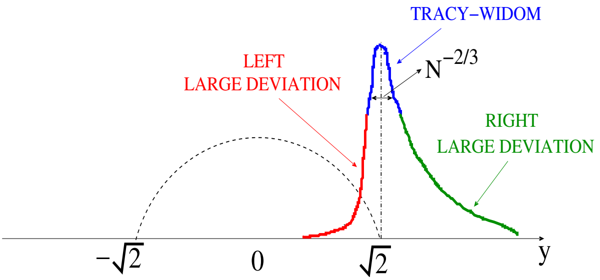
Open problems:

- The exponent η_k and the scaling function $f^{(k)}(z)$ that characterize the typical fluctuations of $x_{\max}$ around its mean, for $k \neq 0, -1$
- Other observables: order statistics, gap statitics, full-counting statistics etc. for $k \neq 0, -1$

$x_{\max}$ for the Log-gas $k \to 0^+$



$$P_0\left[y_{\max} = w, N\right] \approx \begin{cases} \exp\left[-\beta\, N^2\, \Phi_-(w,0)\right] & \text{for} \quad \sqrt{2} - w \gtrsim O(1) \\ N^{2/3} f^{(0)}_{\mathrm{GOE}}\left(N^{2/3}\left(w - \sqrt{2}\right)\right) & \text{for} \quad |w - \sqrt{2}| \lesssim O(N^{-2/3}) \\ \exp\left[-\beta\, N\, \Phi_+(w,0)\right] & \text{for} \quad w - \sqrt{2} \gtrsim O(1). \end{cases}$$

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D. S. Dean & S.M., PRL, 97, 160201 (2006)

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Pushed to pulled transition

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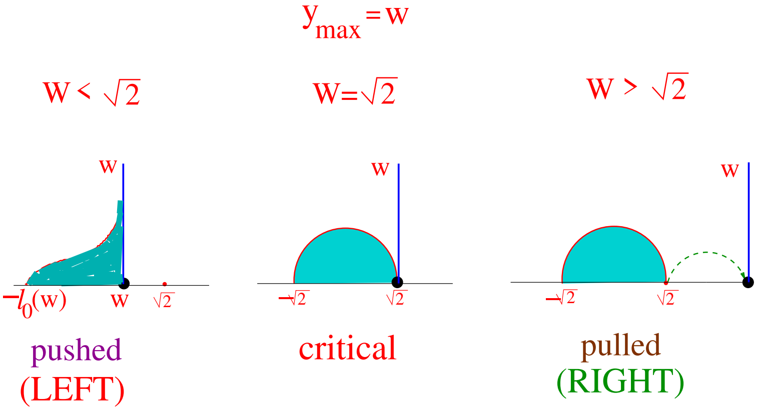
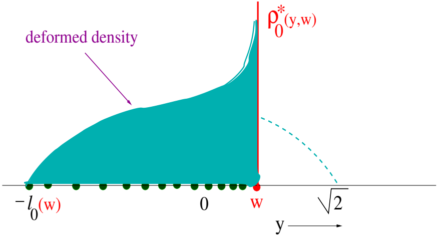


W=$\sqrt{2}$ ⟶ CRITICAL POINT

3-rd order phase transition



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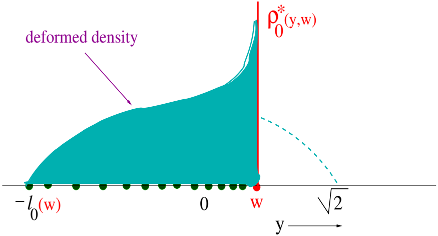

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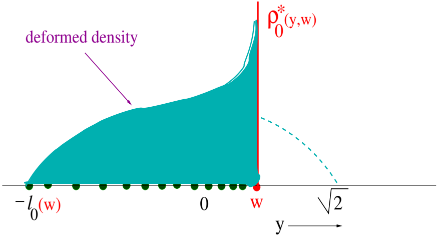

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3-rd phase transition $\Rightarrow$ Gauge theory, KPZ, liquid crystal, spin glasses,

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