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On quantum statistics transmutation via flux attachment

Nicolas Rougerie
UMPA
CNRS & ENS Lyon

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Joint work with Gaultier Lambert and Douglas Lundholm

Quantum mechanics/quantum statistics

- ▶ Many-body non-relativistic quantum mechanics in \mathbb{R}^d
- ▶ Action of many-body Schrödinger Hamiltonian on $L^2(\mathbb{R}^{dN})$

$$H_N = \sum_{j=1}^N -\Delta_{\mathbf{x}_j} + V(\mathbf{x}_j) + \sum_{1 \leq i < j \leq N} W(\mathbf{x}_i - \mathbf{x}_j)$$

- ▶ Add symmetry : **indistinguishable particles** \rightarrow **quantum statistics**
- ▶ Two possible classes for fundamental particles

$$\Psi_N(x_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_N) = \Psi_N(x_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_i, \dots, x_N) \quad \text{Bosons}$$

$$\Psi_N(x_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, x_N) = -\Psi_N(x_1, \dots, \mathbf{x}_j, \dots, \mathbf{x}_i, \dots, x_N) \quad \text{Fermions}$$

- ▶ Bosons, work on $L^2_{\text{sym}}(\mathbb{R}^{dN}) = \bigotimes_{\text{sym}} L^2(\mathbb{R}^d)$
- ▶ Fermions, work on $L^2_{\text{asym}}(\mathbb{R}^{dN}) = \bigotimes_{\text{asym}} L^2(\mathbb{R}^d)$

This talk: how to change quantum statistics by playing with magnetic fields in 2D.

Magnetic fields, change of gauge (in 2D)

- ▶ External magnetic field $B(\mathbf{x}) \in \mathbb{R}$, vector potential $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^2$

$$\text{curl } \mathbf{A} = \partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = B$$

- ▶ Change canonical momentum $\mathbf{p} = -i\nabla_{\mathbf{x}}$ to $\mathbf{p}_A = -i\nabla_{\mathbf{x}} + \mathbf{A}$
- ▶ Schrödinger operator becomes

$$H_N = \sum_{j=1}^N (-i\nabla_{\mathbf{x}_j} + \mathbf{A}(\mathbf{x}_j))^2 + V(\mathbf{x}_j) + \sum_{1 \leq i < j \leq N} w(\mathbf{x}_i - \mathbf{x}_j)$$

- ▶ Gauge invariance: joint change of wave-function/vector potential

$$\Psi_N \rightarrow \Psi_N \prod_{j=1}^N e^{i\varphi(\mathbf{x}_j)} \text{ and } \mathbf{A} \rightarrow \mathbf{A} - \nabla\varphi$$

- ▶ Indeed, in density/phase representation $\psi = \sqrt{\rho}e^{i\phi}$

$$\langle \psi, (-i\nabla + \mathbf{A})^2 \psi \rangle = \int |\nabla\sqrt{\rho}|^2 + \int \rho |\nabla\phi + \mathbf{A}|^2$$

⇒ in 2D, one can switch between bosonic and fermionic statistics.

Switching between bosonic and fermionic statistics

- ▶ Many-body vector potential and magnetic field (**Aharonov-Bohm flux**)

$$\mathbf{A}(\mathbf{x}_j) := \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \quad B(\mathbf{x}_j) = 2\pi \sum_{k \neq j} \delta_{\mathbf{x}_j = \mathbf{x}_k}$$

- ▶ Many-body phase factor (arg = angle of a vector)

$$\varphi_1(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{1 \leq j < k \leq N} \arg(\mathbf{x}_j - \mathbf{x}_k)$$

- ▶ If Ψ_N is bosonic, $\Phi_N = e^{i\varphi_1} \Psi_N$ is fermionic and

$$\left\langle \Psi_N, \sum_{j=1}^N -\Delta_{\mathbf{x}_j} \Psi_N \right\rangle = \left\langle \Phi_N, \sum_{j=1}^N (-i\nabla_{\mathbf{x}_j} + \mathbf{A}(\mathbf{x}_j))^2 \Phi_N \right\rangle$$

2D bosons \leftrightarrow 2D fermions with attached (integer) magnetic flux
(and vice-versa)

Emergent fractional statistics/anyons

- ▶ **What if the attached flux is fractional** ? Leinaas-Myrheim, Wilczek, Goldin-Menikoff-Sharp 77-82 invent ANYONS
- ▶ Pick any $\alpha \in \mathbb{R}$ and set

$$\mathbf{A}_\alpha(\mathbf{x}_j) := \alpha \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \quad B(\mathbf{x}_j) = 2\pi\alpha \sum_{k \neq j} \delta_{\mathbf{x}_j = \mathbf{x}_k}$$

- ▶ Many-body phase factor (arg = angle of a vector)

$$\varphi_\alpha(\mathbf{x}_1, \dots, \mathbf{x}_N) = \alpha \sum_{1 \leq j < k \leq N} \arg(x_j - x_k)$$

- ▶ If Ψ_N is bosonic/fermionic, set

$$\Phi_N = e^{i\varphi_\alpha} \Psi_N$$
$$\left\langle \Phi_N, \sum_{j=1}^N -\Delta_{\mathbf{x}_j} \Phi_N \right\rangle = \left\langle \Psi_N, \sum_{j=1}^N (-i\nabla_{\mathbf{x}_j} - \mathbf{A}_\alpha(\mathbf{x}_j))^2 \Psi_N \right\rangle$$

- ▶ FORMALLY, Φ_N has exotic exchange phase

$$\Phi_N(x_1, \dots, x_i, \dots, x_j, \dots, x_N) = e^{i\pi\alpha} \Phi_N(x_1, \dots, x_j, \dots, x_i, \dots, x_N)$$

attaching non-integer magnetic flux \leftrightarrow realization of non-standard quantum statistics

Attaching magnetic flux: quantum Hall physics

Main ingredients (for the original effect in Gallium-Arsenide heterostructures)

- ▶ A gas of electrons trapped in a 2D plane
- ▶ A strong perpendicular magnetic field b
- ▶ The Pauli principle and/or interparticle repulsion

Usual approximation

- ▶ All particles forced to ground eigenspace of one-body kinetic energy

$$\left(-i\nabla - bx^\perp\right)^2$$

- ▶ **Lowest Landau Level** in symmetric gauge

$$\text{LLL} := \left\{ \psi \in L^2(\mathbb{R}^2), \psi(x) = f(z)e^{-\frac{b}{2}|z|^2}, f \text{ analytic} \right\}$$

- ▶ Replace $L^2_{\text{asym}}(\mathbb{R}^{2N})$ by $\bigotimes_{\text{asym}}^N \text{LLL}$
- ▶ Family of **Laughlin states** (1983): with m odd integer

$$\Psi^{(m)}(z_1, \dots, z_N) = c_m \prod_{1 \leq i < j \leq N} (z_i - z_j)^m e^{-\frac{b}{2}|z_j|^2}$$

- ▶ $m = 1$: free fermions, locally filled Landau level
- ▶ $m > 1$: account for interactions in a partially filled Landau level

Facts in quantum Hall physics

- ▶ Excitations/impurities in Laughlin-like states

$$\Psi_{\text{qh}}^{(m)}(z_1, \dots, z_N) = \Psi^{(m)}(z_1, \dots, z_N) \prod_{j=1}^N \prod_{k=1}^K (z_j - a_k)$$

a_1, \dots, a_K location of **QUASI-HOLES**

- ▶ Are the effective charge carriers (Laughlin 83)
- ▶ Carry charge $1/m$ (Laughlin 83, Saminadayar-Glattli-Jin-Etienne 97)
- ▶ Have anyon statistics $1/m$ (Halperin, Arovas-Schrieffer-Wilcek 84, Bartolomei-et al-Fève, Nakamura-et al-Manfra 20)



Probes of such behavior ? Mathematical results in this direction ?

Thought experiment in bath-tracers system

- ▶ N fermions in the LLL (bath), n quantum impurities of charge q (tracers)
- ▶ Joint Hilbert space $\mathfrak{H}^{n \oplus N} = L^2(\mathbb{R}^{2n}) \otimes \text{LLL}^{\otimes \text{asym} N}$
- ▶ Joint Hamiltonian with strong repulsive interactions, large $g > 0$

$$H_{n \oplus N} := g \sum_{k=1}^N \sum_{j=1}^n W(x_k - y_j) + \sum_{j=1}^n \left(-i \nabla_{y_j} - q b y_j^\perp \right)^2 + \dots$$

- ▶ Natural trial state “cancels” interactions ($\mathbf{w}, \mathbf{z} = y, x$ in complex notation)

$$\Psi_\Phi(\mathbf{w}; \mathbf{z}) := \Phi(\mathbf{y}) c(\mathbf{w}) \left(\prod_{j=1}^n \prod_{k=1}^N (w_j - z_k)^p \right) \left(\prod_{1 \leq k < \ell \leq N} (z_k - z_\ell)^m \prod_{k=1}^N e^{-b|z_k|^2/2} \right)$$

Statement (Lundholm-NR 16)

Large $b = N$ limit. Let $\mathbf{A}^{\text{tot}}(y_j)$ the total vector potential

$$\mathbf{A}^{\text{tot}}(y_j) := - \left(q - \frac{p}{m} \right) b y_j^\perp - \frac{p^2}{m} \sum_{\ell \neq j} \frac{(y_j - y_\ell)^\perp}{|y_j - y_\ell|^2}.$$

$$\int_{\mathbb{R}^{2(n+N)}} \left| \left(-i \nabla_{y_j} - q b y_j^\perp \right) \Psi_\Phi \right|^2 = 2b \frac{p}{m} + \int_{\mathbb{R}^{2n}} \left| \left(-i \nabla_{y_j} + \mathbf{A}^{\text{tot}}(y_j) \right) \Phi \right|^2 + \text{Errors}$$

Mathematical challenges

1. Prove energy estimate rigorously (appropriate smallness of Errors)
2. Consider contact/delta interactions $W = \delta_0$ (makes sense on $\mathfrak{H}^{n \oplus N}$). Is the trial state a true quasi-mode ?
3. Optimality of the trial state: depends on the **FQH-spectral gap conjecture** (work by Nachtergaele-Warzel-Young)

This talk: deal with **1** in the special case $m = p = 1$.

$$\Psi_{\Phi}(\mathbf{w}; \mathbf{z}) := \Phi(\mathbf{y})c(\mathbf{w}) \left(\prod_{j=1}^n \prod_{k=1}^N (w_j - z_k) \right) \left(\prod_{1 \leq k < \ell \leq N} (z_k - z_{\ell}) \prod_{k=1}^N e^{-b|z_k|^2/2} \right)$$

- ▶ Bath = free fermions in LLL. Scale $b = N$ (no loss)

$$\left| \Psi^{(1)}(z_1, \dots, z_N) \right|^2 = c_1^2 \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 e^{-N|z_j|^2}$$

- ▶ Law for eigenvalues of random matrices from Ginibre's ensemble.
- ▶ Moments of associated characteristic polynomial

$$\int_{\mathbb{R}^{2N}} \left| \prod_{j=1}^n \prod_{k=1}^N (w_j - z_k) \right|^2 \left| \Psi^{(1)}(z_1, \dots, z_N) \right|^2$$

Statistics transmutation

- ▶ Turn bosons into fermions (and vice-versa)

$$\Phi(y_1, \dots, y_n) = \left(\prod_{1 \leq i < j \leq n} e^{i \arg(y_i - y_j)} \right) \tilde{\Phi}(y_1, \dots, y_n)$$

- ▶ Set $b = N$, fix κ large enough. Droplet: extension of the fermionic bath

$$\mathcal{D}_n := \{(y_1, \dots, y_n) \in \mathbb{R}^{2n} : |y_j| \leq 1 - \kappa \sqrt{\frac{\log N}{N}}\}$$

- ▶ No merging set

$$\mathcal{D}_n^\emptyset := \{(y_1, \dots, y_n) \in \mathcal{D}_n : |y_i - y_j| \geq 2\kappa \sqrt{\frac{\log N}{N}}\}$$

Theorem (Lambert-Lundholm-NR, 22)

Let $\tilde{\Phi}$ have support in \mathcal{D}_n . Assume $|\tilde{\Phi}(\mathbf{y})| \leq C|y_i - y_j|$.

$$\int_{\mathbb{R}^{2(n+N)}} \left| (-i\nabla_{y_j} - qb y_j^\perp) \Psi_\Phi \right|^2 = 2b + \int_{\mathbb{R}^{2n}} \left| (-i\nabla_{y_j} - (q-1)by_j^\perp) \tilde{\Phi} \right|^2$$

$$+ O\left(\kappa^4 \frac{(\log N)^3}{N}\right) + O\left(\kappa^2 \frac{(\log N)^{3/2}}{N^{1/2}} \left(\int_{\mathcal{D}_n \setminus \mathcal{D}_n^\emptyset} \left| (-i\nabla_{y_j} - (q-1)by_j^\perp) \tilde{\Phi} \right|^2 \right)^{1/2}\right)$$

Illustration: equal charges, bosonic tracers

- ▶ Joint Hamiltonian, smooth potential W

$$H_{n \oplus N} := g \sum_{k=1}^N \sum_{j=1}^n \delta_0(x_k - y_j) + \sum_{j=1}^n \left(-i \nabla_{y_j} - qb y_j^\perp \right)^2 + W(y_1, \dots, y_n)$$

on $\mathfrak{H}^{n+N} = L^2_{\text{sym}}(\mathbb{R}^{2n}) \otimes \text{LLL}^{\otimes \text{asym} N}$

- ▶ **Effective Hamiltonian for tracers**

$$H_n^{\text{eff}} = \sum_{j=1}^n -\Delta_{y_j} + W(y_1, \dots, y_n)$$

- ▶ **Fermionic ground state energy** (\mathcal{D}_α disk of radius α)

$$E^{\text{eff}}(n) := \inf \left\{ \langle U_n | H_n^{\text{eff}} | U_n \rangle, U_n \in H_0^1(\mathcal{D}_\alpha^n), \int_{\mathcal{D}_\alpha^n} |U_n|^2 = 1, U_n \text{ anti-symmetric} \right\}$$

Corollary (Lambert-Lundholm-NR 22)

Set $q = 1$ and let $E(n \oplus N)$ be the lowest eigenvalue of $H_{n \oplus N}$. Fix $\alpha < 1$. We have

$$E(n \oplus N) \leq 2nb + E^{\text{eff}}(n) + C_n \frac{(\log N)^3}{N}$$

in the limit $b = N \rightarrow \infty$.

Emerging potentials and the plasma analogy

- ▶ Direct/lengthy calculation (probably folklore in adiabatic theory)

$$\int_{\mathbb{R}^{2(n+N)}} \left| \left(-i\nabla_{y_j} - by_j^\perp \right) \Psi_\Phi \right|^2 = \int_{\mathbb{R}^{2n}} \left| \left(-i\nabla_{y_j} - by_j^\perp + \mathcal{A}_j \right) \Phi \right|^2 + \int_{\mathbb{R}^{2n}} |\Phi|^2 \mathcal{V}_j$$

- ▶ Using LLL properties (analyticity)

$$\mathcal{A}_j(\mathbf{w}) = \frac{1}{2} \nabla_{w_j}^\perp \log c^{-2}(\mathbf{w}) \quad \mathcal{V}_j(\mathbf{w}) = \frac{1}{2} \Delta_{w_j} \log c^{-2}(\mathbf{w})$$

- ▶ $c(\mathbf{w}) = L^2$ normalization constant

$$c(\mathbf{w})^{-2} := \int_{\mathbb{C}^N} \left| \left(\prod_{j=1}^n \prod_{k=1}^N (w_j - z_k) \right) \left(\prod_{1 \leq k < \ell \leq N} (z_k - z_\ell) \prod_{k=1}^N e^{-\frac{b}{2} |z_k|^2 / 2} \right) \right|^2 dz$$

- ▶ Partition function of Gibbs ensemble for **fictitious 2D Coulomb system**

$$\mu(\mathbf{z}) = \frac{1}{\mathcal{Z}(\mathbf{w})} \exp(-bH(\mathbf{w}; \mathbf{z})) \text{ probability measure for all } \mathbf{w}$$

$$H(\mathbf{w}; \mathbf{z}) = \sum_{j=1}^N |z_j|^2 + 2b^{-1} \sum_{1 \leq i < j \leq N} \log \frac{1}{|z_i - z_j|} + 2b^{-1} \sum_{j=1}^N \sum_{k=1}^n \log \frac{1}{|z_j - w_k|}.$$

Screening in the 2D one-component plasma

- ▶ In the regime of interest, there should be screening in the plasma
- ▶ **Local charge neutrality**

$$Q^{\text{tot}} = \sum_{j=1}^N \delta_{z_j} + \sum_{j=1}^n \delta_{w_j} - \frac{1}{\pi} \mathbb{1}_{D(0,R)} \simeq 0$$

- ▶ Free energy satisfies (completing a square + neglecting entropy)

$$\begin{aligned} F(\mathbf{w}) &= -b^{-1} \log \mathcal{Z}(\mathbf{w}) = 2b^{-1} \log c(\mathbf{w}) \\ &\simeq - \sum_{j=1}^n |w_j|^2 - 2b^{-1} \sum_{1 \leq i < j \leq N} \log \frac{1}{|w_i - w_j|} + C. \end{aligned}$$

where C does not depend on \mathbf{w} .

- ▶ Differentiating yields the desired expressions

Issue: Validity and precision of screening heuristics far from obvious in general plasmas (Bauerschmidt-Bourgade-Nikula-Yau, Leblé-Serfaty, Armstrong-Serfaty, Leblé, Serfaty 2015-2022+...)

Using the determinantal structure

- ▶ Ginibre correlation kernel/LLL projector

$$K_N(z, x) := \sum_{0 \leq j < N} \frac{b^{j+1}}{\pi j!} z^j \bar{x}^j e^{-b(|z|^2 + |x|^2)/2}$$

- ▶ Interest in the characteristic polynomial lead to

Theorem (G. Lambert + other authors)

For any $\mathbf{w} \in \mathbb{C}^n$ with $w_1 \neq \dots \neq w_n$, we have

$$c(\mathbf{w})^{-2} = b^{-\frac{(n+N)(n+N-1)}{2}} \pi^{n+N} \frac{(n+N-1)! \prod_{j=1}^n e^{b|w_j|^2}}{n! |\Delta(\mathbf{w})|^2} \det_{n \times n} [K_{n+N}(w_i, w_j)]$$

with the Vandermonde determinant.

$$\Delta(\mathbf{w}) := \prod_{1 \leq i < j \leq n} (w_j - w_i)$$

- ▶ Proof: reduced density matrices of free fermions, Wick's theorem.
- ▶ Use: reduce to control derivatives of

$$\log \left(\det_{n \times n} [K_{n+N}(w_i, w_j)] \right)$$

Correlation kernel

$$\mathbf{M}(\mathbf{w}) := [\pi K_{n+N}(w_i, w_j)]_{n \times n} \quad \Upsilon(\mathbf{w}) := \pi^n \mathfrak{b}^{-n} \det \mathbf{M}(\mathbf{w})$$

Goal: $\Upsilon \simeq 1$ for large $N = \mathfrak{b}$

- ▶ **Jacobi's formula for the derivatives.** Control of

$$\begin{aligned} \partial_{w_j} \log \Upsilon(\mathbf{w}) &= \text{Tr} \left(\mathbf{M}(\mathbf{w})^{-1} \partial_{w_j} \mathbf{M}(\mathbf{w}) \right) \\ 4\Delta_{w_j} \log \Upsilon(\mathbf{w}) &= 4\text{Tr} \left(\mathbf{M}(\mathbf{w})^{-1} \Delta_{w_j} \mathbf{M}(\mathbf{w}) \right) \\ &\quad - \text{Tr} \left(\frac{1}{\mathbf{M}(\mathbf{w})} \partial_{w_j} \mathbf{M}(\mathbf{w}) \frac{1}{\mathbf{M}(\mathbf{w})} \partial_{w_j} \mathbf{M}(\mathbf{w}) \right) \end{aligned}$$

- ▶ **Correlation kernel for large N**

$$\begin{aligned} K_N(z, w) &\simeq K_\infty^{(\mathfrak{b})}(z, w) = \frac{\mathfrak{b}}{\pi} e^{-\frac{\mathfrak{b}}{2}(|z|^2 + |w|^2 - 2z\bar{w})} \\ |K_\infty^{(\mathfrak{b})}(z, w)| &= \frac{\mathfrak{b}}{\pi} e^{-\mathfrak{b}|z-w|^2/2} \end{aligned}$$

- ▶ Hence, for separated w_i, w_j

$$\mathbf{M}(\mathbf{w}) \simeq N^{-1} \times (\text{Identity matrix})$$

Dealing with tracer encounters

- ▶ **Single quasi-holes merging:** $w_1 \simeq w_2$ but all other points “far away”
- ▶ With $\mathbf{M}_{3-n} \simeq$ Identity

$$\mathbf{M} = \begin{pmatrix} M_{12} & V \\ V^* & \mathbf{M}_{3-n} \end{pmatrix}$$

- ▶ Schur's complement formula

$$\partial_{w_1} \log \Upsilon(\mathbf{w}) = \text{Tr} \left(\frac{1}{\mathbf{M}_{12} - V\mathbf{M}_{3-n}^{-1}V^*} \partial_{w_1} \left(\mathbf{M}_{12} - V\mathbf{M}_{3-n}^{-1}V^* \right) \right)$$

- ▶ Explicit calculations to handle contributions from \mathbf{M}_{12}
- ▶ **Multiple tracer encounters/merging:** global rough bounds from integral formulae for emerging potentials e.g.

$$\mathcal{A}_j(\mathbf{w}) = N\Im \left(\begin{pmatrix} 1 \\ i \end{pmatrix} \left(\int_{\mathbb{C}} \frac{\Upsilon(\mathbf{w}, z)}{\Upsilon(\mathbf{w})} \frac{dz}{\pi(w_j - z)} - \bar{w}_j \right) \right)$$

- ▶ Schur again for $\Upsilon(\mathbf{w}, z)$.

Conclusions

- ▶ Unique to 2D quantum mechanics: one can **represent bosons as fermions using gauge freedom**, and vice-versa
- ▶ Does not seem a very good idea for calculations, but suggests funny mechanisms using magnetic flux attachment
- ▶ **Impurities in quantum-Hall systems** couple peculiarly to magnetic fields
- ▶ Can lead to **modified effective charge and statistics transmutation**
- ▶ We rigorously proved that natural trial state exhibits such an effect
- ▶ Much remains to be done to investigate emergence of trial state
- ▶ Main prospect: the same type of mechanism leads to **exotic quantum statistics (anyons)**
- ▶ In that case, even rigorously computing with trial state seems challenging (non-integrable classical 2D Coulomb system, β -ensemble)

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