Local statistics in the 1d Coulomb gas: extremes, gaps and full-counting statistics

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in collaboration with

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- Satya N. Majumdar (LPTMS, Univ. Paris-Saclay)
- Sanjib Sabhapandit (RRI, Bangalore)

The 1d Coulomb gas: a classical stat. mech. problem

- A. Lenard, Exact Statistical Mechanics of a One-Dimensional System with Coulomb Forces, (1961)
- S. Prager, The One-Dimensional Plasma, (1962)
- R. J. Baxter, Statistical mechanics of a 1d Coulomb system with a uniform charge background, (1963)
- H. Kunz, The 1d classical electron gas, (1974)
- M. Aizenman, P. A. Martin, Structure of Gibbs states of one dimensional Coulomb systems, (1980)
- ...

Local statistics in the 1d Coulomb gas: extremes, gaps and full-counting statistics

- A. Dhar, A. Kundu, S. N. Majumdar, S. Sabhapandit, G. S., Exact extremal statistics in the classical 1d Coulomb gas, Phys. Rev. Lett. 119, 060601 (2017)

- A. Dhar, A. Kundu, S. N. Majumdar, S. Sabhapandit, G. S., Extreme statistics and index distribution in the classical 1d Coulomb gas, J. Phys. A: Math. Theor. 51 295001 (2018)

- A. Flack, S. N. Majumdar, G. S., Gap probability and full counting statistics in the one dimensional one-component plasma, J. Stat. Mech. (2022) 053211

One dimensional (neutral) plasma

- N +ve charges $+q$ (with positions $x_i's$)
  N -ve charges $-q$ (with positions $y_i's$) on $[-L, L]$

- Energy of a configuration with $\gamma \propto q^2$

$$E[\{x_i\},\{y_i\}] = -\gamma \sum_{i \neq j} |x_i - x_j| - \gamma \sum_{i \neq j} |y_i - y_j| + \gamma \sum_{i \neq j} |x_i - y_j|$$

1d Coulomb interaction

- Treat the negative particles as a uniform background with a density $\rho_0 = N/(2L)$ over $[-L, L]$

$$\gamma \sum_{i \neq j} |x_i - y_j| \simeq \gamma\, \rho_0 \sum_{i=1}^{N} \int_{-L}^{L} |x_i - y|\, dy = \gamma\, \rho_0 \sum_{i=1}^{N} (L^2 + x_i^2)$$

effective harmonic potential

- Different types of backgrounds have recently been considered in Chafaï, Garcia-Zelada, Jung (2021)

One dimensional jellium model on the line $L \to \infty$

- Effective energy for the +ve charges

$$E[\{x_i\}] = A\sum_{i=1}^{N} x_i^2 - B\sum_{i\neq j}|x_i - x_j| \quad \text{with} \quad A, B = \mathcal{O}(1)$$

- Typical scale L_N : $\quad x_i = L_N\,\tilde{x}_i \quad$ with $\quad \tilde{x}_i = O(1)$

$$A\sum_{i=1}^{N} x_i^2 \sim A N L_N^2 \qquad \text{vs} \qquad B\sum_{i\neq j}|x_i - x_j| \sim B\,N^2 L_N$$

$$\Longrightarrow L_N = O(N)$$

- Dimensionless energy, setting $A = 1$, $B = \alpha$

$$\beta E[\{x_i\}] = \frac{N^2}{2}\sum_{i=1}^{N}\tilde{x}_i^2 - \alpha\, N\sum_{i\neq j}|\tilde{x}_i - \tilde{x}_j|$$

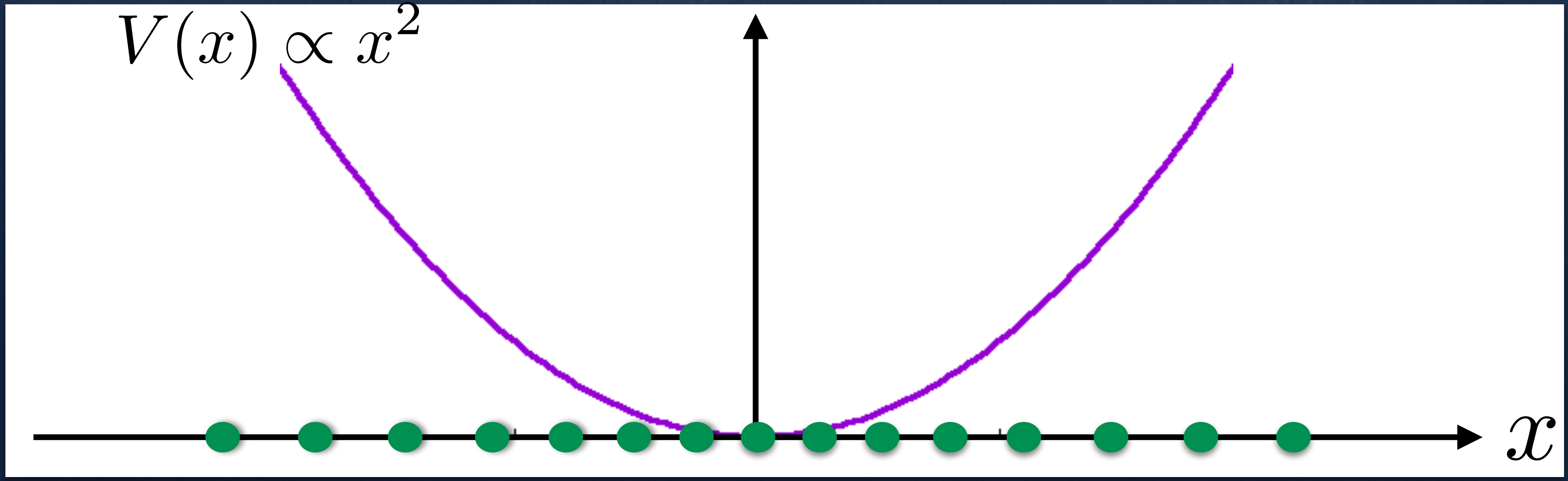
One-dimensional jellium

- One-dimensional jellium model (1d-one component plasma)

$$P(x_1, x_2, \cdots, x_N) = \frac{1}{Z_N} \exp\left[-\beta\, E(x_1, x_2, \cdots, x_N)\right]$$

$$\beta E[\{x_i\}] = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha\, N \sum_{i \neq j} |x_i - x_j|$$

1d Coulomb interaction



One-dimensional jellium: average density

$$\beta E(x_1, \cdots, x_N) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|$$

- Average density: $\rho_N(x) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle$

- Order the positions: $x_{(1)} < x_{(2)} < \cdots < x_{(N)}$

Lenart '61, Baxter '63

$$\beta E(\{x_i\}) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - 2\alpha N \sum_{i>j} (x_{(i)} - x_{(j)})$$

$$= \frac{N^2}{2} \sum_{i=1}^{N} x_{(i)}^2 - 2\alpha N \sum_{i=1}^{N} (2i - N - 1) x_{(i)}$$

$$= \frac{N^2}{2} \sum_{i=1}^{N} \left(x_{(i)} - \frac{2\alpha}{N} (2i - N - 1) \right)^2 + C_N(\alpha)$$

One-dimensional Coulomb gas: average density

- Order the positions: $x_{(1)} < x_{(2)} < \cdots < x_{(N)}$

$$\beta\, E(x_1, \cdots, x_N) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|$$

$$= \frac{N^2}{2} \sum_{i=1}^{N} \left(x_{(i)} - \frac{2\alpha}{N}(2i - N - 1) \right)^2 + C_N(\alpha)$$

- Equilibrium positions: $x_{(i)}^* = \frac{2\alpha}{N}(2i - N - 1) \quad , \quad i = 1, 2, \cdots, N$

⟹ equisplaced positions

leftmost $x_{(1)}^* = -2\alpha \left(1 - \frac{1}{N} \right)$

rightmost $x_{(N)}^* = 2\alpha \left(1 - \frac{1}{N} \right)$

⟹ uniform density

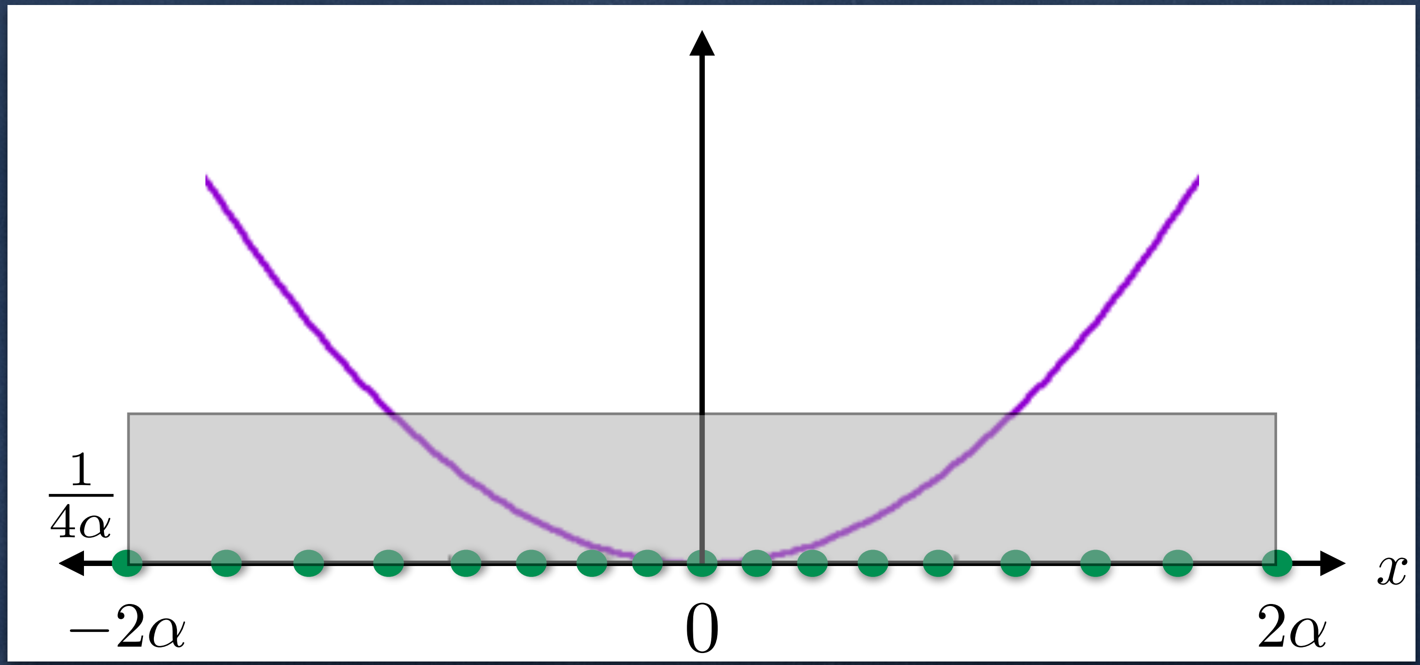
One-dimensional Coulomb gas: average density

$$\beta E(x_1, \cdots, x_N) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|$$

- Average density of particles

$$\rho_N(x) = \frac{1}{N} \sum_{i=1}^{N} \langle \delta(x - x_i) \rangle$$

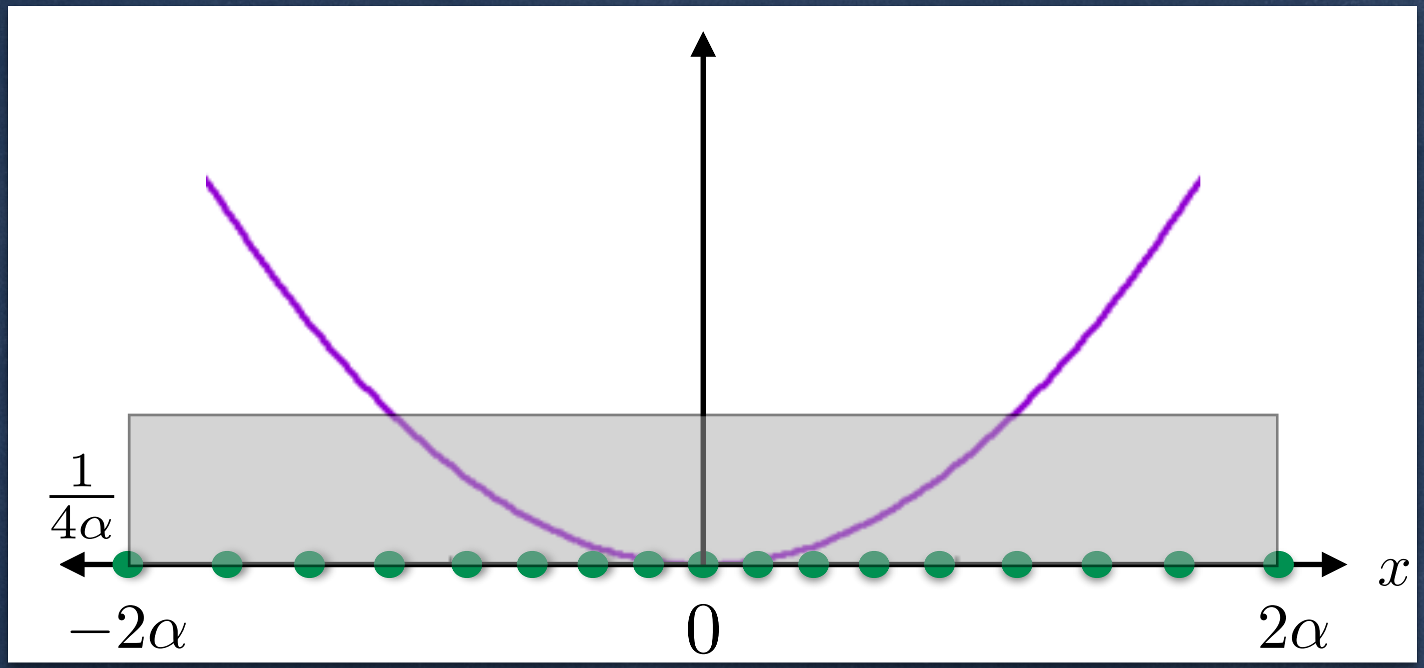
$$\underset{N \to \infty}{\longrightarrow} \rho_\infty(x) = \frac{1}{4\alpha} \ , \ |x| \leq 2\alpha$$



- Quite generic for d-dimensional Coulomb gas + harmonic potential (see e.g. the Ginibre ensemble in d=2)

One-dimensional Coulomb gas

$$\beta E(x_1, \cdots, x_N) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|$$



What about local fluctuations?

This talk

- Fluctuations of the position of the rightmost particle
- Distribution of the gap between two particles
- Full counting statistics

Local fluctuations at the edge of the one-dimensional Coulomb gas

- A. Dhar, A. Kundu, S. N. Majumdar, S. Sabhapandit, G. S., Phys. Rev. Lett. 119, 060601 (2017)

- A. Dhar, A. Kundu, S. N. Majumdar, S. Sabhapandit, G. S., J. Phys. A: Math. Theor. 51 295001 (2018)

Motivations and background: edge universality for Gaussian beta-ensembles of RMT

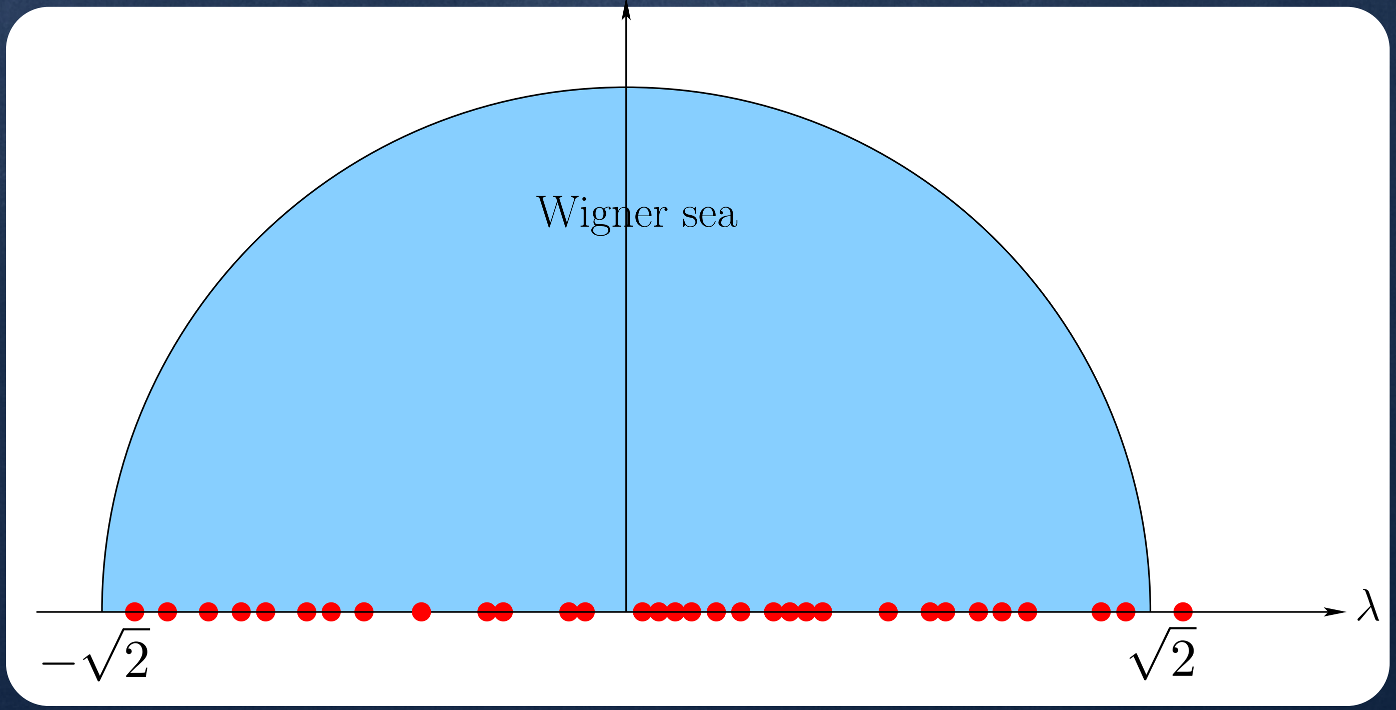
- Dyson 's log-gas at temperature β (Gaussian β-ensemble)

$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\beta\left(N\sum_{i=1}^{N}\frac{x_i^2}{2} - \frac{1}{2}\sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right]$$

- Mean density of eigenvalues

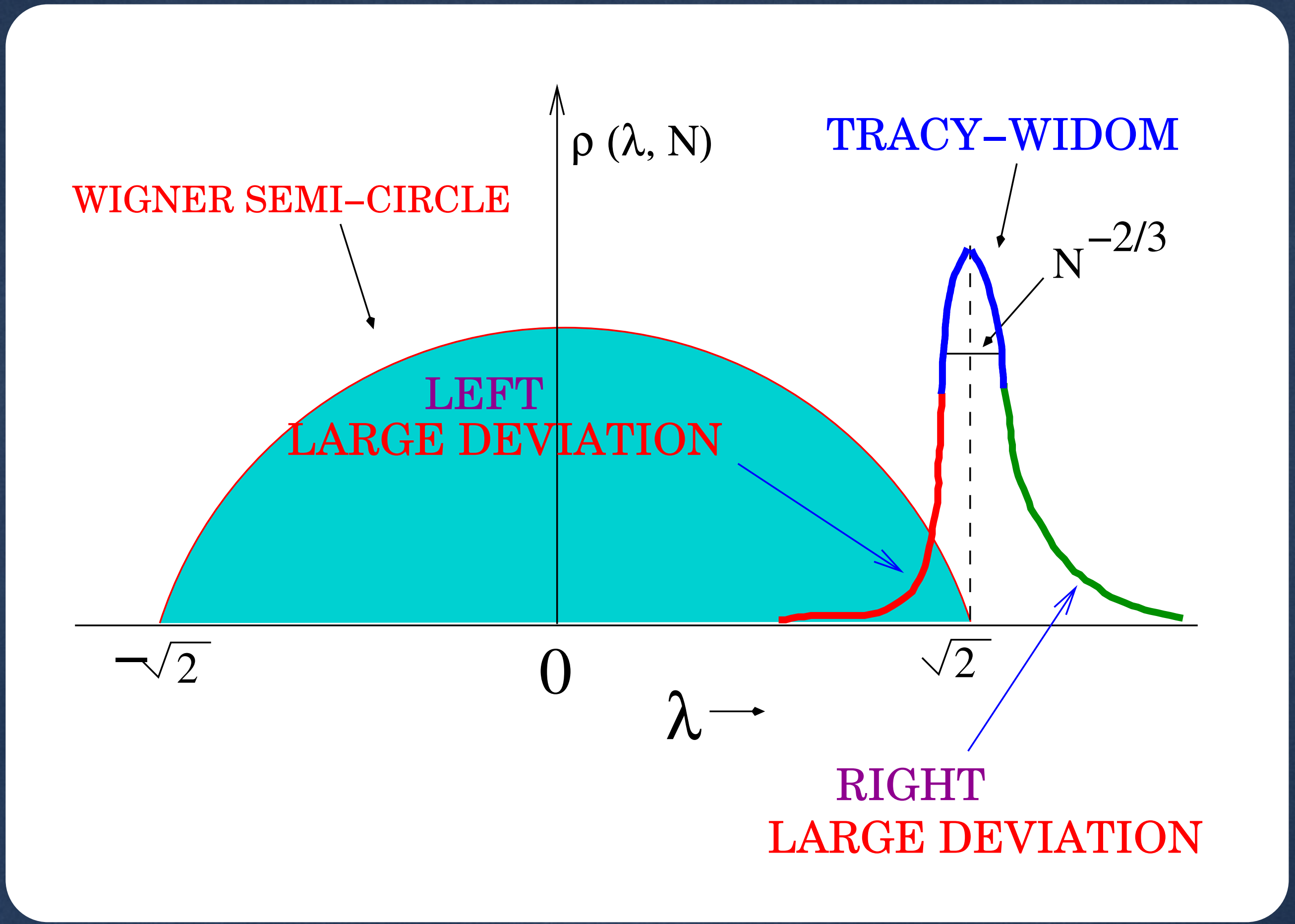
$$\rho_N(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(\lambda-\lambda_i)\rangle$$

$$\underset{N\to\infty}{\longrightarrow} \rho_{\mathrm{SC}}(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}$$



Tracy-Widom distributions for $\lambda_{\max}$

$$\lambda_{\max} = \max_{1 \le i \le N} \lambda_i$$

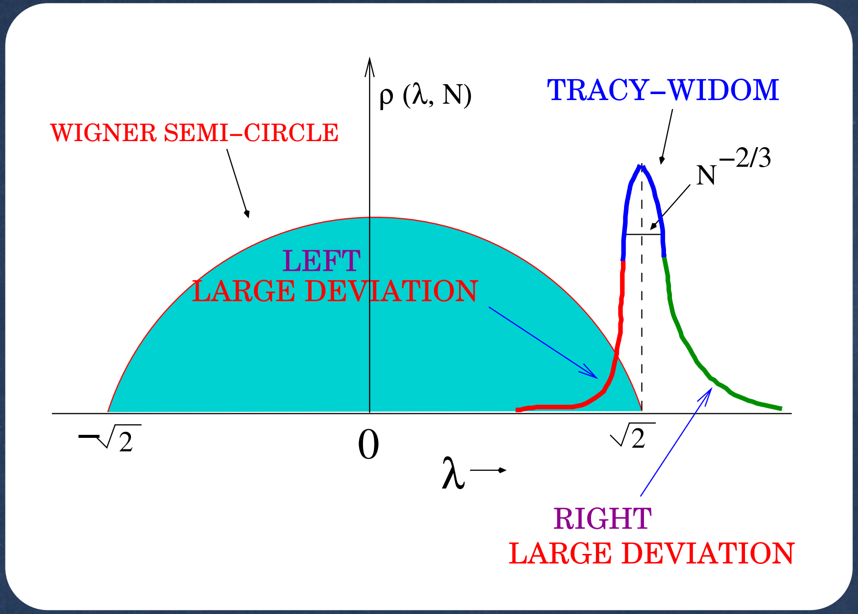


- In the limit $N \to \infty$, $\lambda_{\max} \to \sqrt{2}$
- Tracy-Widom distributions and their generalizations

$$\Pr.[\lambda_{\max} \le w] \longrightarrow \mathcal{F}_\beta(\sqrt{2}(w-\sqrt{2})N^{2/3})$$

Tracy & Widom '94

Tracy-Widom distributions for $\lambda_{\max}$



- For $\beta = 1, 2, 4$ explicit expression in terms of a Painlevé transcendent

Tracy & Widom '94, '96

$$q''(x) = xq(x) + 2q^3(x)\ ,\ q(x) \underset{x \to \infty}{\sim} \mathrm{Ai(x)}$$

For instance for $\beta = 2$: $$\mathcal{F}_2(x) = \exp\left[-\int_x^\infty (s-x)\, q^2(s)\, ds\right]$$

- For generic values of $\beta > 0$: stochastic Airy operator

Dumitriu, Edelman '02
Ramirez, Rider, Virag '11

Motivations and background: edge universality for beta-ensembles of RMT

- Dyson 's log-gas at temperature β

$$P(\lambda_1, \cdots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\beta \left(N \sum_{i=1}^{N} V(\lambda_i) - \frac{1}{2} \sum_{i \neq j} \ln|\lambda_i - \lambda_j| \right)\right]$$

- Edge properties are universal for a wide class of confining potentials $V(\lambda)$ such that the density has a finite support and vanishes as a square-root at the edge (« regular » potential)

Krishnapur, Rider, Virag '13
Bourgade, Erdös, Yau '14

Q: what happens if the interactions are changed ?

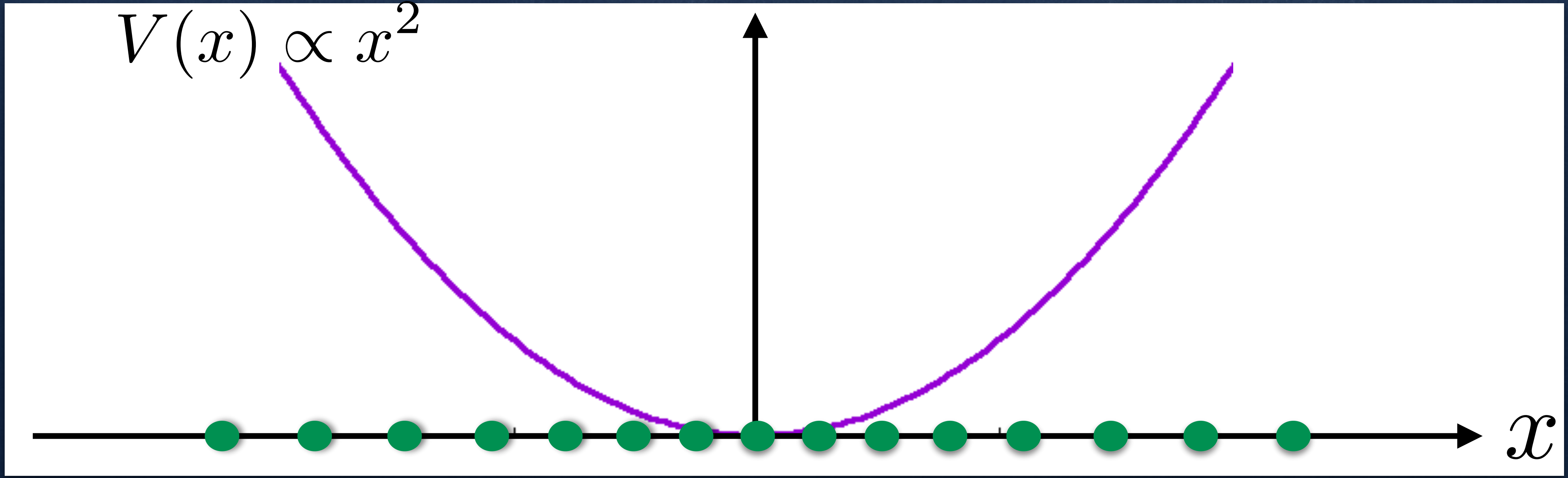
Back to the one-dimensional jellium

- One-dimensional jellium model (1d-one component plasma)

$$P(x_1, x_2, \cdots, x_N) = \frac{1}{Z_N} \exp\left[-\beta\, E(x_1, x_2, \cdots, x_N)\right]$$

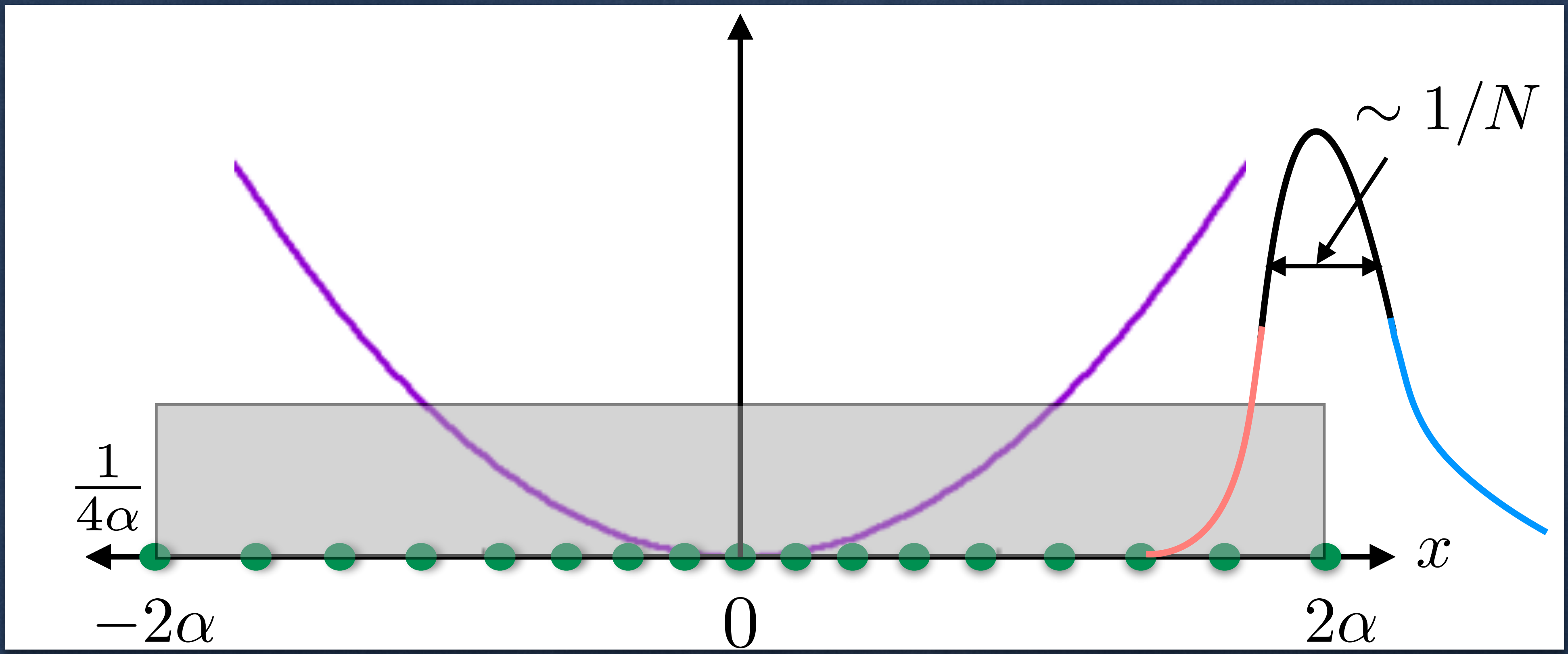
$$\beta E[\{x_i\}] = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha\, N \sum_{i \neq j} |x_i - x_j|$$

1d Coulomb interaction



1d Coulomb gas: fluctuations at the edge

$$x_{\max} = \max_{1 \le i \le N} x_i$$



- In the limit $N \to \infty$, $x_{\max} \to 2\alpha$
- The typical scale of fluctuations of $x_{\max}$ can be obtained via

$$\int_{x_{\max}}^{2\alpha} \rho_\infty(x)\, dx \sim \frac{1}{N} \Longrightarrow 2\alpha - x_{\max} = \mathcal{O}(1/N)$$

1d Coulomb gas: typical fluctuations of $x_{\max}$

$$Q_N(w) = \mathbb{P}(x_{\max} < w)$$

- Limiting form for large N, with $N(w - 2\alpha) = z = \mathcal{O}(1)$

$$Q_N(w) \underset{N \to \infty}{\longrightarrow} F_\alpha(z + 2\alpha)$$

A. Dhar et al. (2017), (2018)

"eigenvalue"

where
$$\frac{dF_\alpha(x)}{dx} = A(\alpha) e^{-x^2/2} F_\alpha(x + 4\alpha)$$

see also Baxter (1963)

with boundary conditions:
$$\lim_{x \to -\infty} F_\alpha(x) = 0$$
$$\lim_{x \to +\infty} F_\alpha(x) = 1$$ and $$F_\alpha(x) \geq 0 \, , \, \forall x$$

1d Coulomb gas: typical fluctuations of $x_{\max}$

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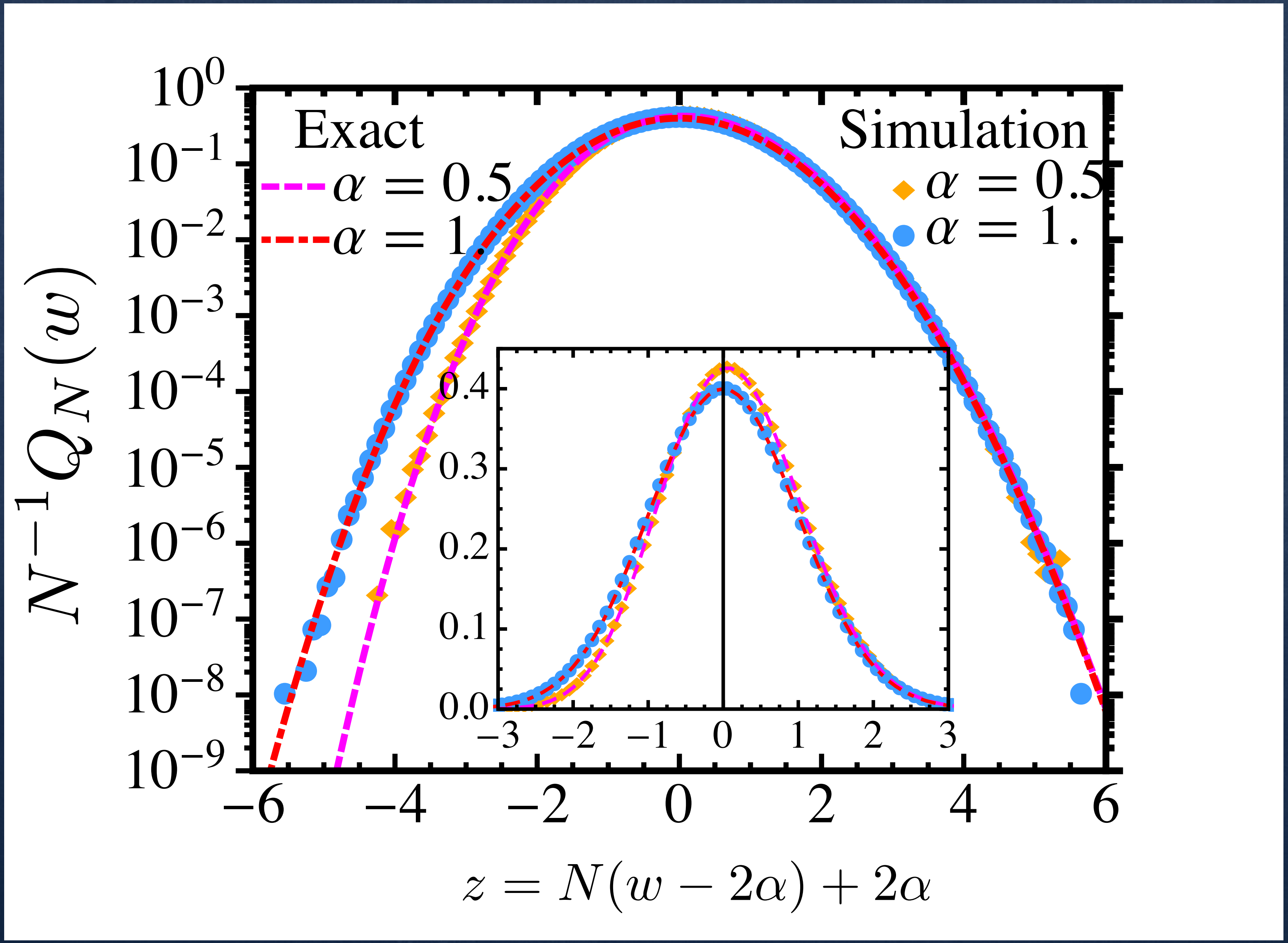
asymptotics
$$F'_\alpha(x) \sim \begin{cases} \exp[-|x|^3/(24\alpha) + \mathcal{O}(x^2)] \,, & x \to -\infty \\ \exp[-x^2/2 + \mathcal{O}(x)] \,, & x \to +\infty \end{cases}$$

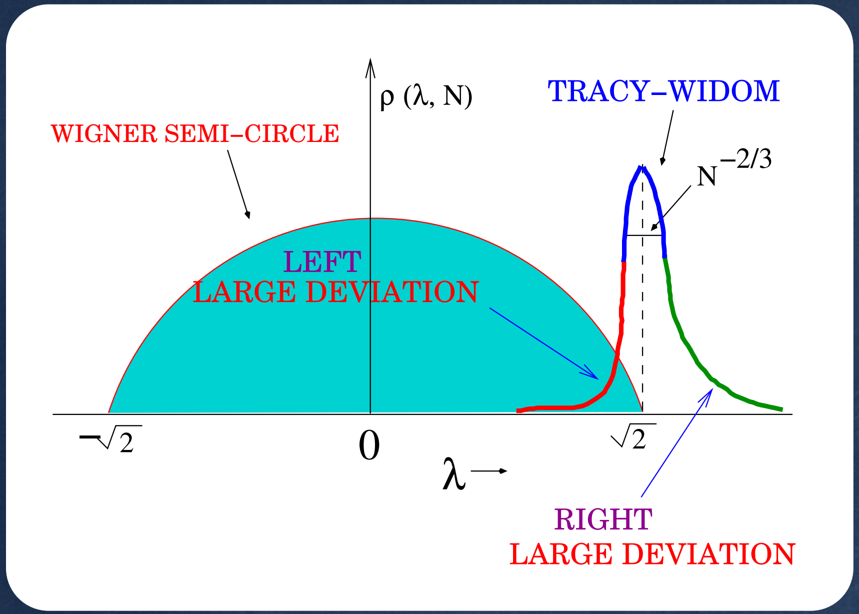
1d Coulomb gas: typical fluctuations of $x_{\max}$

Numerical simulations



A. Dhar et al. (2017), (2018)

Comparison with Tracy-Widom (TW) distributions



Tails of the TW distributions

$$\mathcal{F}'_\beta(x) \approx \begin{cases} \exp\left[-\dfrac{\beta}{24}|x|^3\right], & x \to -\infty \\ \exp\left[-\dfrac{2\beta}{3}x^{3/2}\right], & x \to +\infty, \end{cases}$$

- By contrast our results for the 1d-Coulomb gas yield

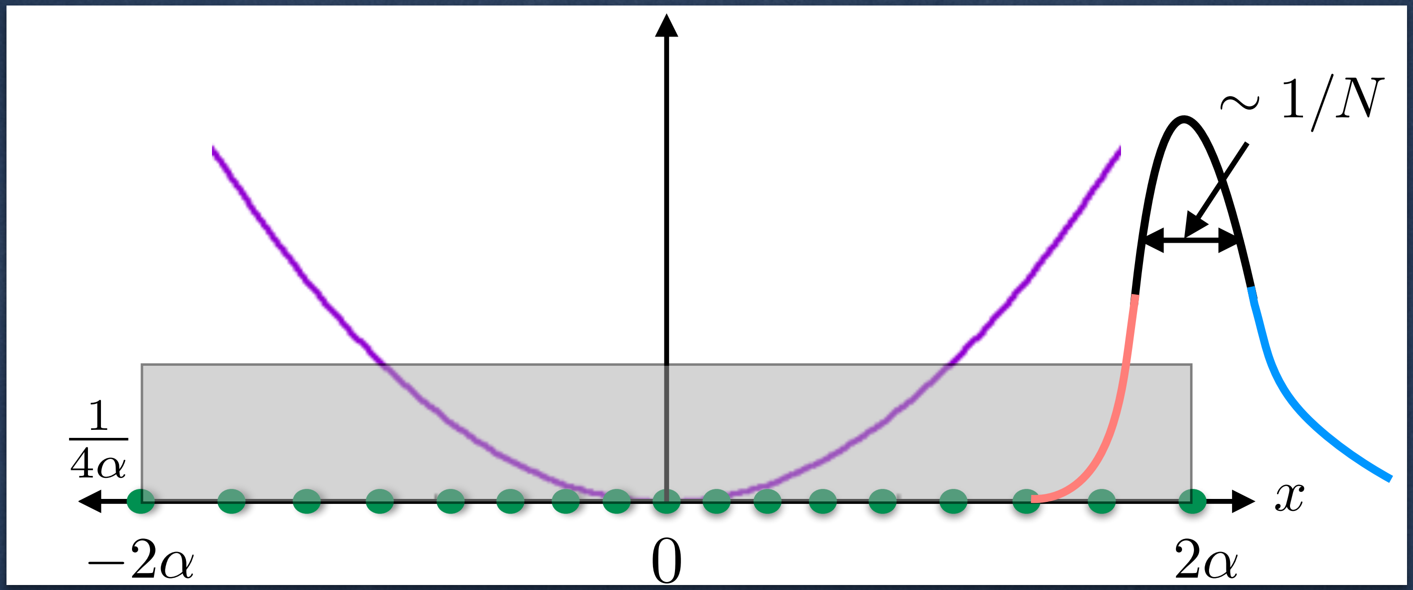
$$F'_\alpha(x) \sim \begin{cases} \exp[-|x|^3/(24\alpha) + \mathcal{O}(x^2)], & x \to -\infty \\ \exp[-x^2/2 + \mathcal{O}(x)], & x \to +\infty \end{cases}$$

⇒ the left tail is « Tracy-Widom like » while the right tail is different

- One can also consider different background charges, which modifies the right tail   Chafaï, Garcia-Zelada, Jung (20121)

1d Coulomb gas: large deviations of $x_{\max}$

What about the fluctuations for $|2\alpha - w| \gg 1/N$?



$$Q_N(w) \sim \begin{cases} e^{-N^3 \Phi_-(w) + \mathcal{O}(N^2)} & , \quad 0 < 2\alpha - w = \mathcal{O}(1) \\ F_\alpha[N(w-2\alpha)+2\alpha] & , \quad |2\alpha - w| = \mathcal{O}(1/N) \\ 1 - e^{-N^2 \phi_+(w) + \mathcal{O}(N)} & , \quad 0 < w - 2\alpha = \mathcal{O}(1) \end{cases}$$

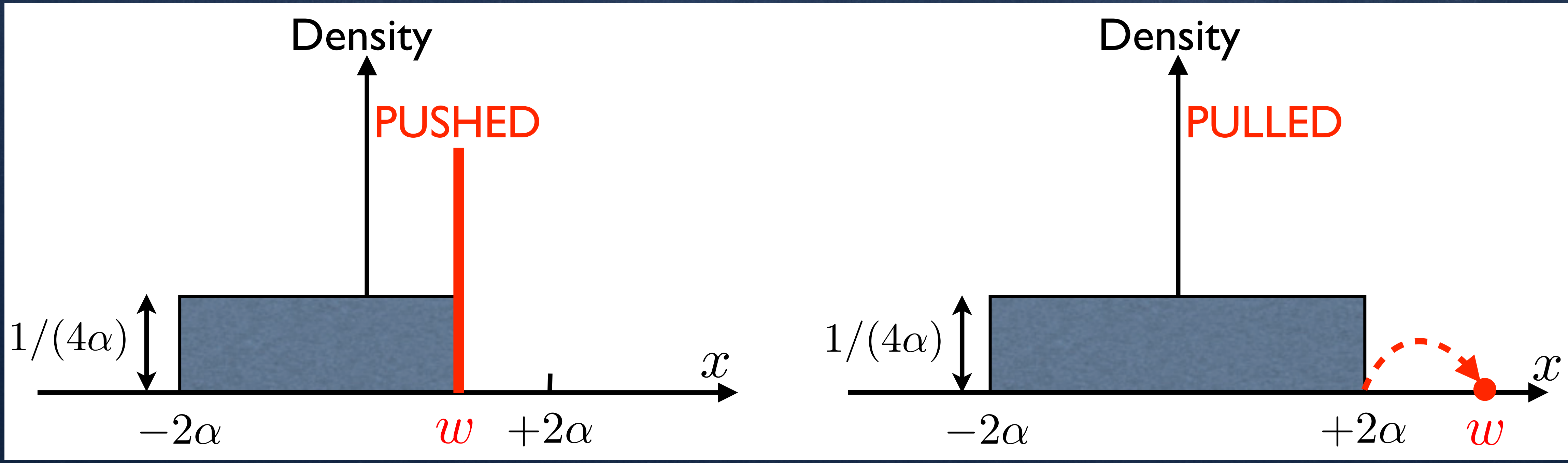
A. Dhar et al. (2017), (2018), see also S. N. Majumdar 's talk

1d Coulomb gas: large deviations of $x_{\max}$

$$Q_N(w) \sim \begin{cases} e^{-N^3\Phi_-(w)+\mathcal{O}(N^2)} & , \quad 0 < 2\alpha - w = \mathcal{O}(1) \\ F_\alpha[N(w-2\alpha)+2\alpha] & , \quad |2\alpha - w| = \mathcal{O}(1/N) \\ 1 - e^{-N^2\phi_+(w)+\mathcal{O}(N)} & , \quad 0 < w - 2\alpha = \mathcal{O}(1) \end{cases}$$

Left tail

Right tail



1d Coulomb gas: large deviations of $x_{\max}$

$$Q_N(w) \sim \begin{cases} e^{-N^3\Phi_-(w)+\mathcal{O}(N^2)} & , \quad 0 < 2\alpha - w = \mathcal{O}(1) \\ F_\alpha[N(w-2\alpha)+2\alpha] & , \quad |2\alpha - w| = \mathcal{O}(1/N) \\ 1 - e^{-N^2\phi_+(w)+\mathcal{O}(N)} & , \quad 0 < w - 2\alpha = \mathcal{O}(1) \end{cases}$$

Left tail

$$\Phi_-(w) = \begin{cases} \frac{(2\alpha-w)^3}{24\alpha} , & -2\alpha \le w \le 2\alpha \\ \frac{w^2}{2} + \frac{2}{3}\alpha^2 , & w \le -2\alpha \end{cases}$$

Right tail

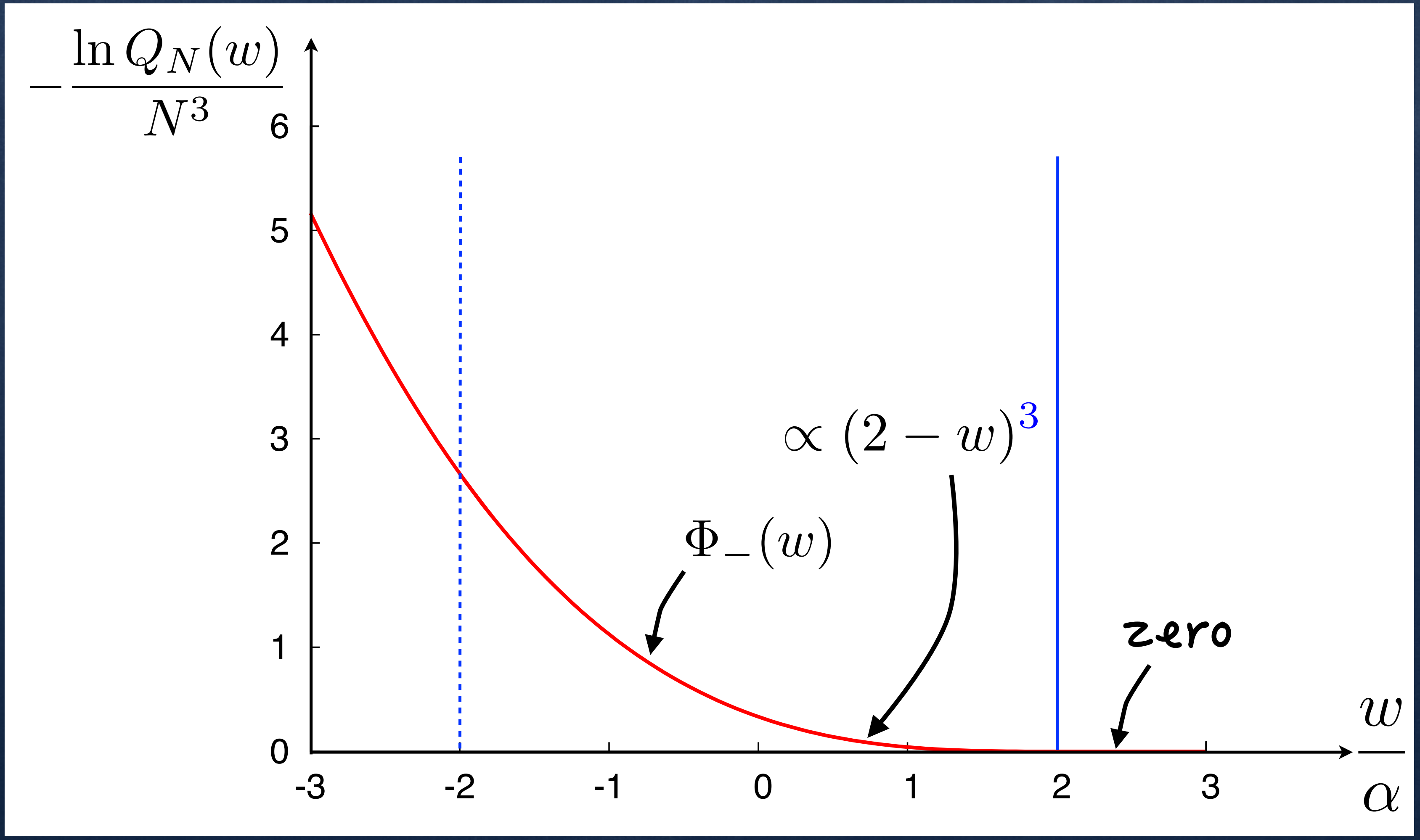
$$\Phi_+(w) = \frac{(w-2\alpha)^2}{2} , \quad w > 2\alpha$$

⟹ Third order phase transition at $w = 2\alpha$

see also S. N. Majumdar, G. S. (2014)

1d Coulomb gas: large deviations of $x_{\max}$

$$Q_N(w) \sim \begin{cases} e^{-N^3\Phi_-(w)+\mathcal{O}(N^2)} , & 0 < 2\alpha - w = \mathcal{O}(1) \\ F_\alpha[N(w-2\alpha)+2\alpha] , & |2\alpha - w| = \mathcal{O}(1/N) \\ 1 - e^{-N^2\phi_+(w)+\mathcal{O}(N)} , & 0 < w - 2\alpha = \mathcal{O}(1) \end{cases}$$



Sketch of the derivation (1/3)

$$Q_N(w) = \mathbb{P}(x_{\max} < w)$$

- Write it as a ratio of two partition functions $Q_N(w) = \frac{Z_N(w)}{Z_N(w \to \infty)}$

$$Z_N(w) = \int_{-\infty}^{w} dx_1 \cdots \int_{-\infty}^{w} dx_N \, \mathrm{e}^{-E(x_1, \cdots, x_N)} \quad \text{(with } \beta = 1\text{)}$$

where $E(x_1, \cdots, x_N) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|$

- Trick: order the positions $x_1 < x_2 < \cdots < x_N$ Lenart '61, Baxter '63

$$Z_N(w) = N! \int_{-\infty}^{w} dx_1 \cdots \int_{-\infty}^{w} dx_N \, \mathrm{e}^{-E(x_1, \cdots, x_N)} \prod_{j=2}^{N} \Theta(x_j - x_{j-1})$$

and get rid of the absolute values in the interaction term

Sketch of the derivation (2/3)

$$Z_N(w) = N! \int_{-\infty}^{w} dx_1 \cdots \int_{-\infty}^{w} dx_N \, \mathrm{e}^{-E(x_1,\cdots,x_N)} \prod_{j=2}^{N} \Theta(x_j - x_{j-1})$$

- For the positions $x_1 < x_2 < \cdots < x_N$ the energy is

$$E(x_1, \cdots, x_N) = \frac{N^2}{2} \sum_{i=1}^{N} x_i^2 - \alpha N \sum_{i \neq j} |x_i - x_j|$$

$$= \frac{N^2}{2} \sum_{i=1}^{N} \underbrace{\left(x_i - \frac{2\alpha}{N}(2i - N - 1) \right)}_{\epsilon_i}^2 + C_N(\alpha)$$

- Change of variable $x_i \longrightarrow \epsilon_i$

$$Z_N(w) = N! \, D_\alpha \left(N \left(w - \frac{2\alpha}{N}(N-1) \right), N \right)$$

$$D_\alpha(x, N) = \int_{-\infty}^{x} d\epsilon_N \int_{-\infty}^{\epsilon_N + 4\alpha} d\epsilon_{N-1} \ldots \int_{-\infty}^{\epsilon_2 + 4\alpha} d\epsilon_1 \, \mathrm{e}^{-\frac{1}{2} \sum_{i=1}^{N} \epsilon_i^2}$$

Sketch of the derivation (3/3)

$$Q_N(w) \equiv F_\alpha(x,N) = \frac{D_\alpha(x,N)}{D_\alpha(\infty,N)} \quad , \quad x = N\left(w - \frac{2\alpha}{N}(N-1)\right)$$

$$D_\alpha(x,N) = \int_{-\infty}^{x} d\epsilon_N \int_{-\infty}^{\epsilon_N+4\alpha} d\epsilon_{N-1} \ldots \int_{-\infty}^{\epsilon_2+4\alpha} d\epsilon_1 \, \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{N}\epsilon_i^2}$$

- Recursion relation

$$\frac{d\,F_\alpha(x,N)}{d\,x} = \frac{D_\alpha(\infty,N-1)}{D_\alpha(\infty,N)}\,\mathrm{e}^{-\frac{x^2}{2}} F_\alpha(x+4\alpha,N-1)$$

where $D_\alpha(\infty,N)$ is the partition function of a short-range interacting particle system: its free energy is extensive

$$D_\alpha(\infty,N) \sim [A(\alpha)]^{-N}$$

1d Coulomb gas: typical fluctuations of $x_{\max}$

$$Q_N(w) = \mathbb{P}(x_{\max} < w)$$

- Limiting form for large N, with $N(w - 2\alpha) = z = \mathcal{O}(1)$

$$Q_N(w) \underset{N\to\infty}{\longrightarrow} F_\alpha(z + 2\alpha)$$

A. Dhar et al. (2017), (2018)

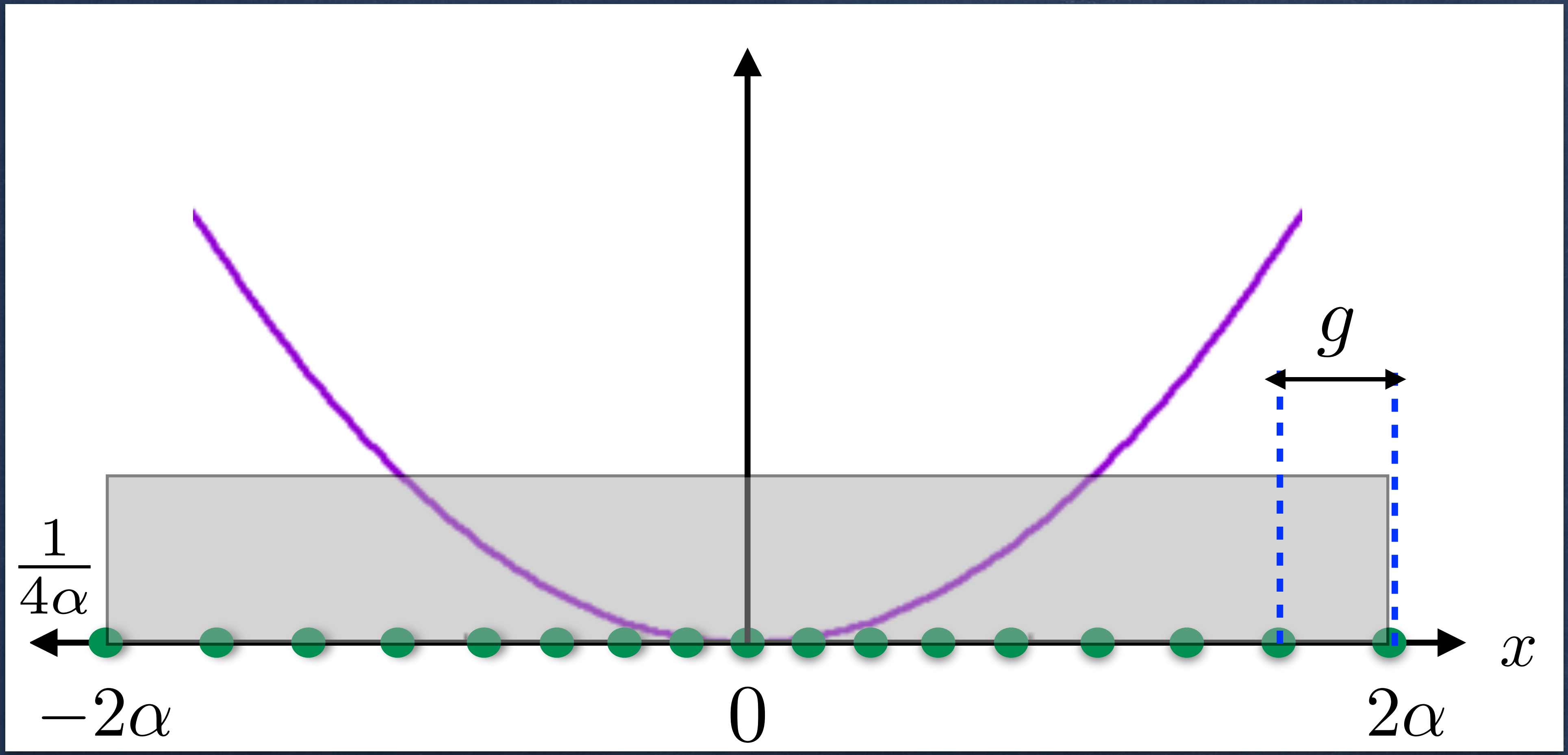
eigenvalue

where
$$\frac{dF_\alpha(x)}{dx} = A(\alpha)e^{-x^2/2}F_\alpha(x + 4\alpha)$$

see also Baxter (1963)

with boundary conditions:
$$\lim_{x\to-\infty} F_\alpha(x) = 0$$
$$\lim_{x\to+\infty} F_\alpha(x) = 1$$
and $F_\alpha(x) \geq 0 \ , \ \forall x$

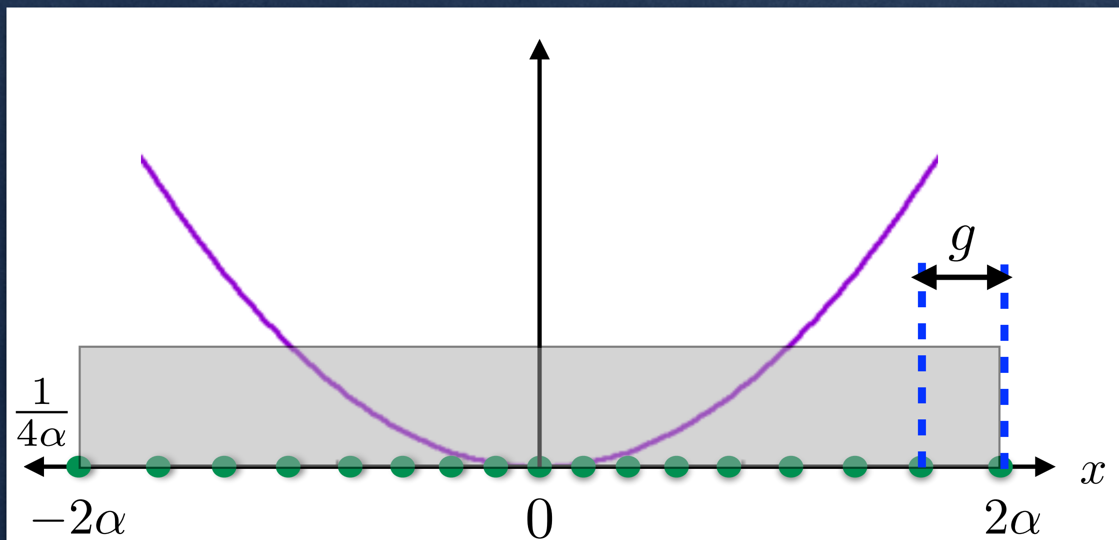
More on the fluctuations at the edge: gap statistics



$$P_{\rm gap,edge}(g,N) \sim \begin{cases} N h_\alpha(N\,g) & , \quad g = \mathcal{O}(1/N) \\ \\ {\rm e}^{-N^2 \Psi_{\rm edge}(g)} & , \quad g = \mathcal{O}(1) \end{cases}$$

A. Dhar et al. (2017), (2018)

More on the fluctuations at the edge: gap statistics



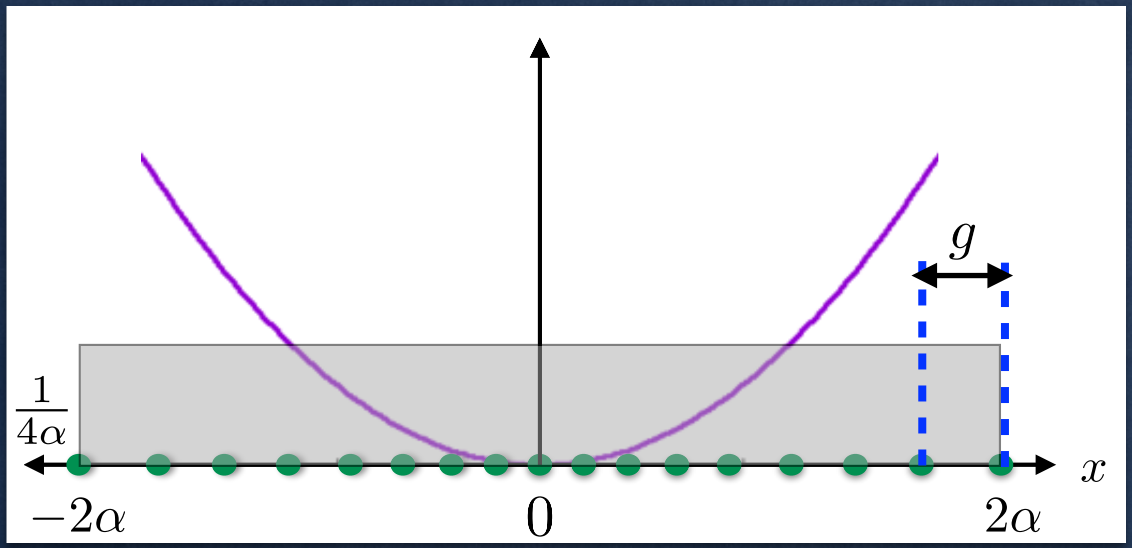
$$P_{\text{gap,edge}}(g,N) \sim \begin{cases} N h_\alpha(N\,g) & , \quad g = \mathcal{O}(1/N) \\ \mathrm{e}^{-N^2 \Psi_{\text{edge}}(g)} & , \quad g = \mathcal{O}(1) \end{cases}$$

- Typical fluctuations $g = \mathcal{O}(1/N)$

$$h_\alpha(z) = A(\alpha) \int_{-\infty}^{\infty} dy\,(y+z-4\alpha)\mathrm{e}^{-\frac{1}{2}(y+z-4\alpha)^2} F_\alpha(y) \ , \ z \geq 0$$

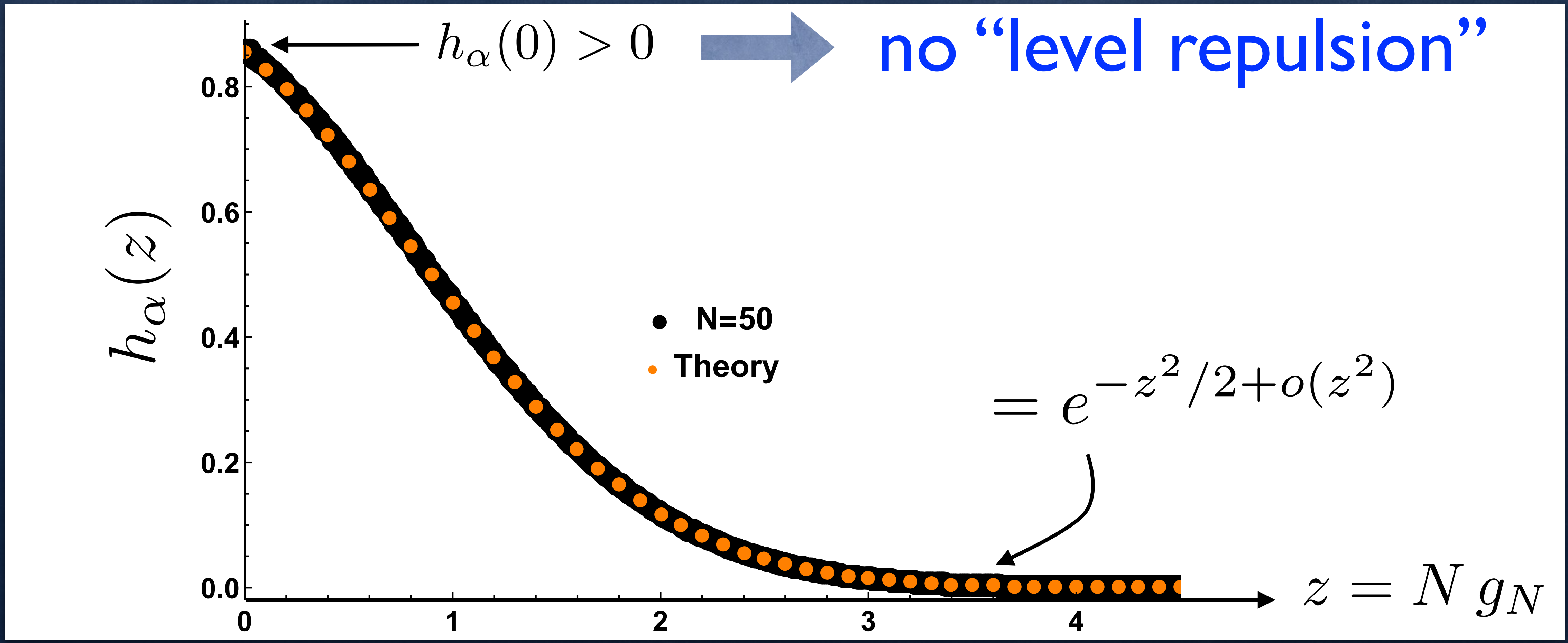
where $$\frac{dF_\alpha(x)}{dx} = A(\alpha) e^{-x^2/2} F_\alpha(x+4\alpha)$$

More on the fluctuations at the edge: gap statistics

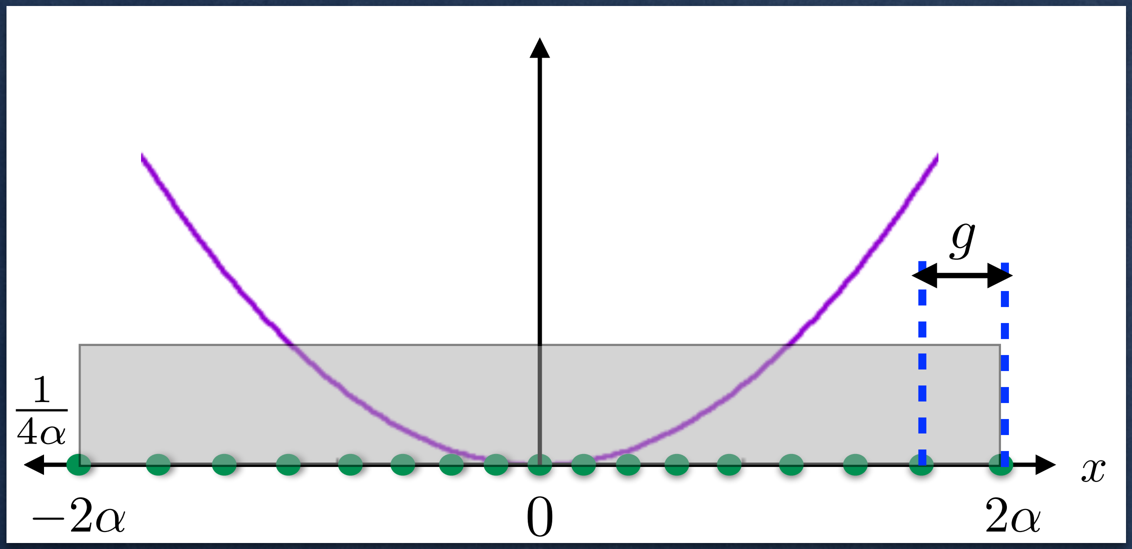


$$P_{\text{gap,edge}}(g,N) \sim \begin{cases} N h_\alpha(N\,g) & , \quad g = \mathcal{O}(1/N) \\ e^{-N^2 \Psi_{\text{edge}}(g)} & , \quad g = \mathcal{O}(1) \end{cases}$$

- Typical fluctuations : $g = \mathcal{O}(1/N)$



More on the fluctuations at the edge: gap statistics



$$P_{\text{gap,edge}}(g, N) \sim \begin{cases} N h_\alpha(N\, g) & , \quad g = \mathcal{O}(1/N) \\ \mathrm{e}^{-N^2 \Psi_{\text{edge}}(g)} & , \quad g = \mathcal{O}(1) \end{cases}$$

- Atypical/large fluctuations : $g = \mathcal{O}(1)$

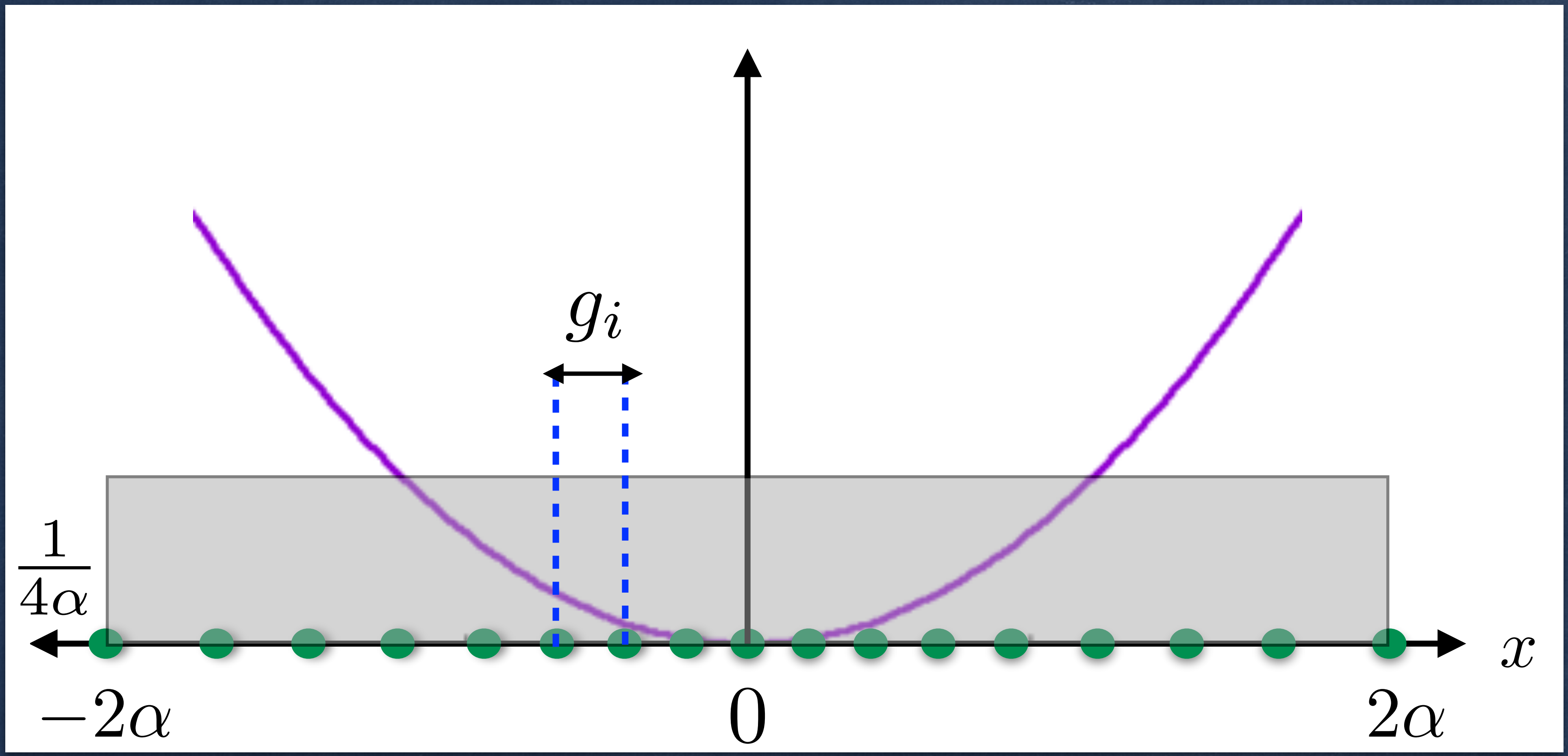
$$\Psi_{\text{edge}}(g) = \frac{g^2}{2}$$

i.e., the prolongation of the tail of $h_\alpha(z)$

Local fluctuations in the bulk of the one-dimensional Coulomb gas

- A. Flack, S. N. Majumdar, G. S., J. Stat. Mech. (2022)

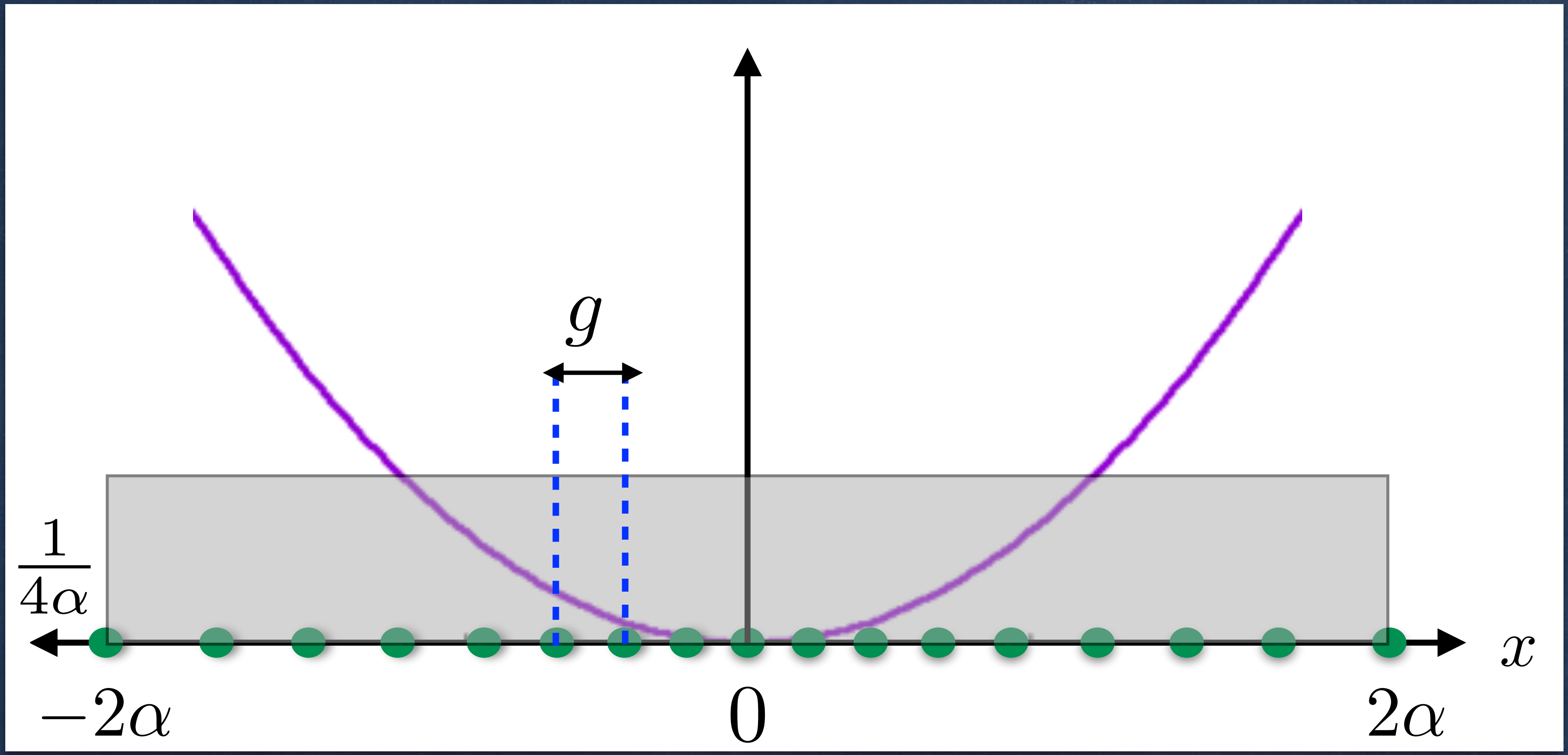
Gap statistics in the bulk



$$x_1 < x_2 < \cdots < x_N$$

- Gap in the bulk: $g_i = x_{i+1} - x_i$ with $i = c\,N\,,\; 0 < c < 1$
- In the bulk, the statistics of g_i is independent of i
- Average value: $\langle g_i \rangle \sim \frac{4\alpha}{N}$

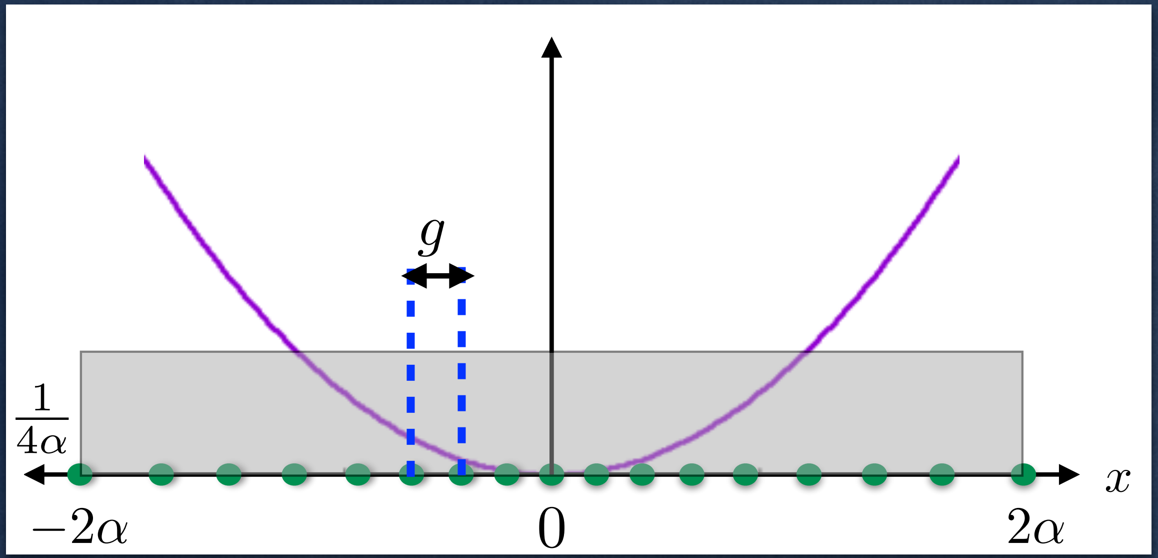
Gap statistics in the bulk



$$P_{\mathrm{gap,bulk}}(g,N) \sim \begin{cases} N\, H_\alpha(g\,N), & g \sim \mathcal{O}(1/N), \\ e^{-N^3 \psi_{\mathrm{bulk}}(g)}, & g \sim \mathcal{O}(1)\ , \end{cases}$$

A. Flack, S. N. Majumdar, G. S., (2022)

Gap statistics in the bulk



$$P_{\rm gap, bulk}(g, N) \sim \begin{cases} N\, H_\alpha(g\, N), & g \sim \mathcal{O}(1/N), \\ e^{-N^3 \psi_{\rm bulk}(g)}, & g \sim \mathcal{O}(1)\,, \end{cases}$$

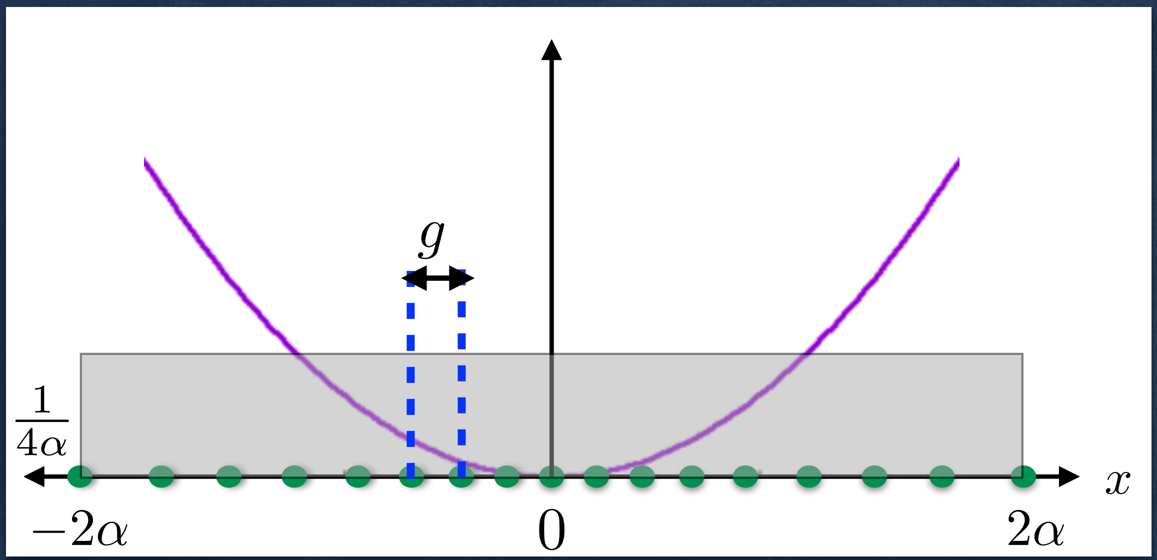
- Typical fluctuations : $g = \mathcal{O}(1/N)$

$$H_\alpha(z) = B(\alpha) \int_{-\infty}^{\infty} dy F_\alpha(y + 4\alpha) F_\alpha(8\alpha - y - z) e^{-y^2/2 - (y+z-4\alpha)^2/2}$$

where $$\frac{dF_\alpha(x)}{dx} = A(\alpha) e^{-x^2/2} F_\alpha(x + 4\alpha)$$

A. Flack, S. N. Majumdar, G. S., (2022)

Gap statistics in the bulk



$$P_{\rm gap,bulk}(g,N) \sim \begin{cases} N\,H_\alpha(g\,N), & g\sim\mathcal{O}(1/N), \\ e^{-N^3\psi_{\rm bulk}(g)}, & g\sim\mathcal{O}(1)\,, \end{cases}$$

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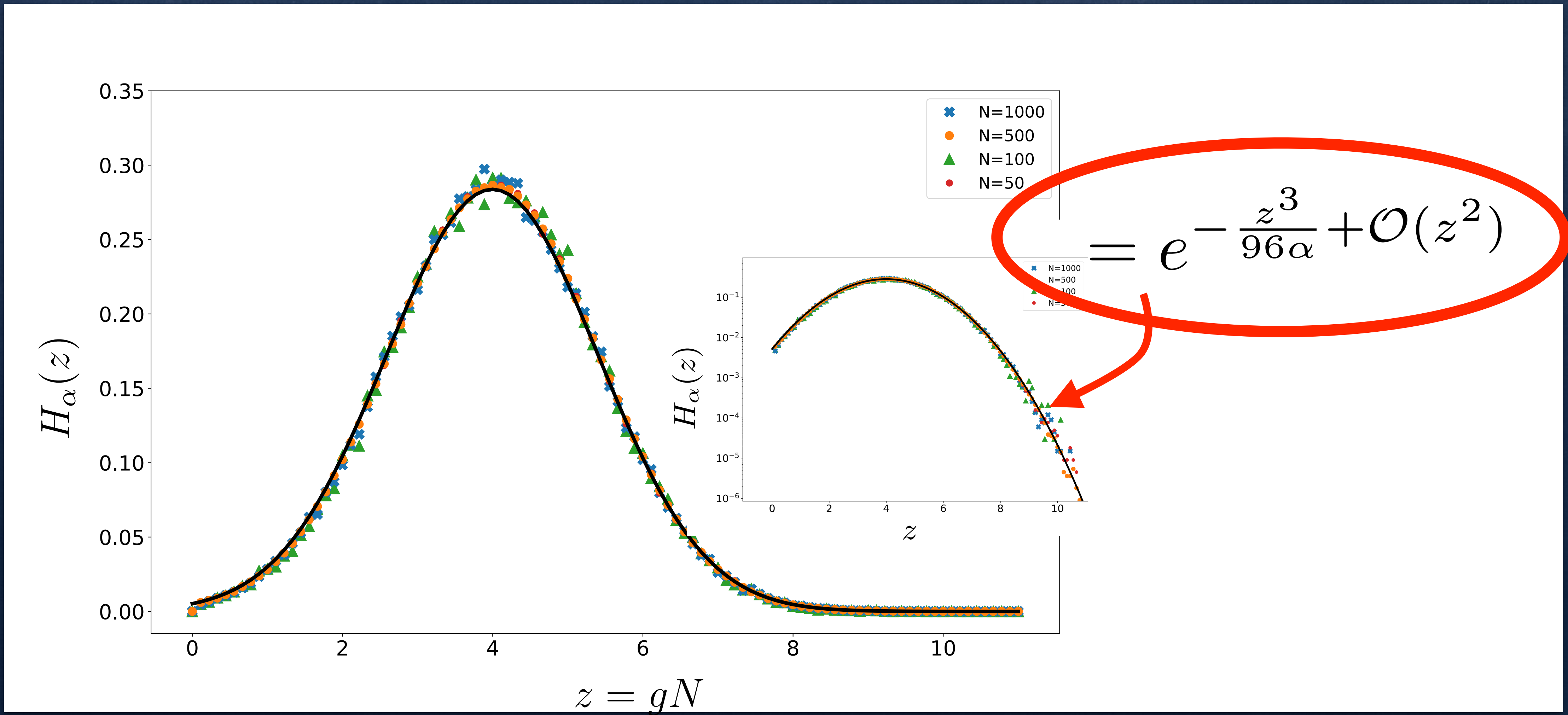
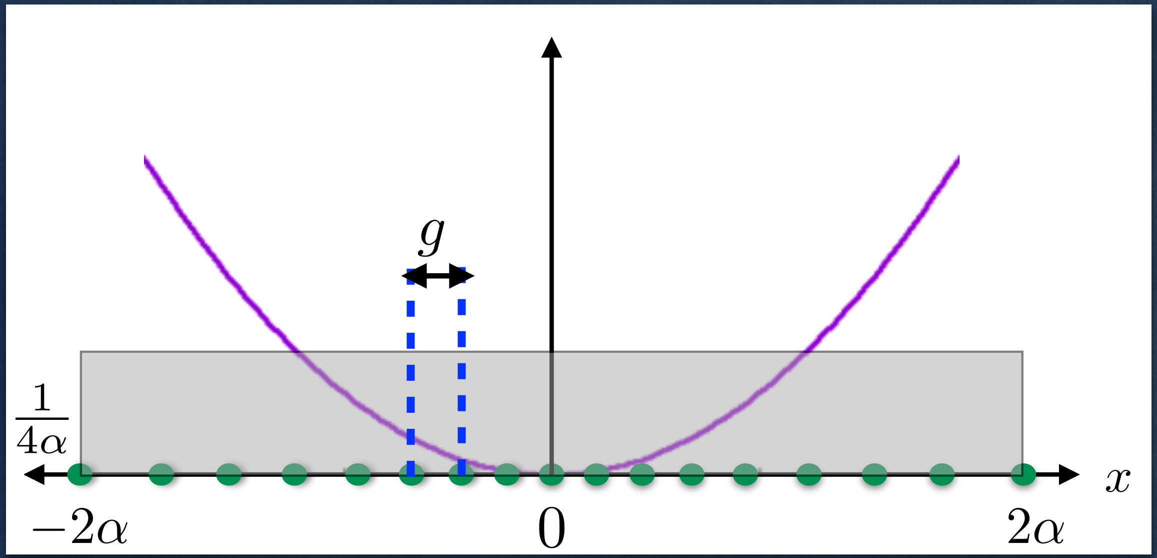


Gap statistics in the bulk



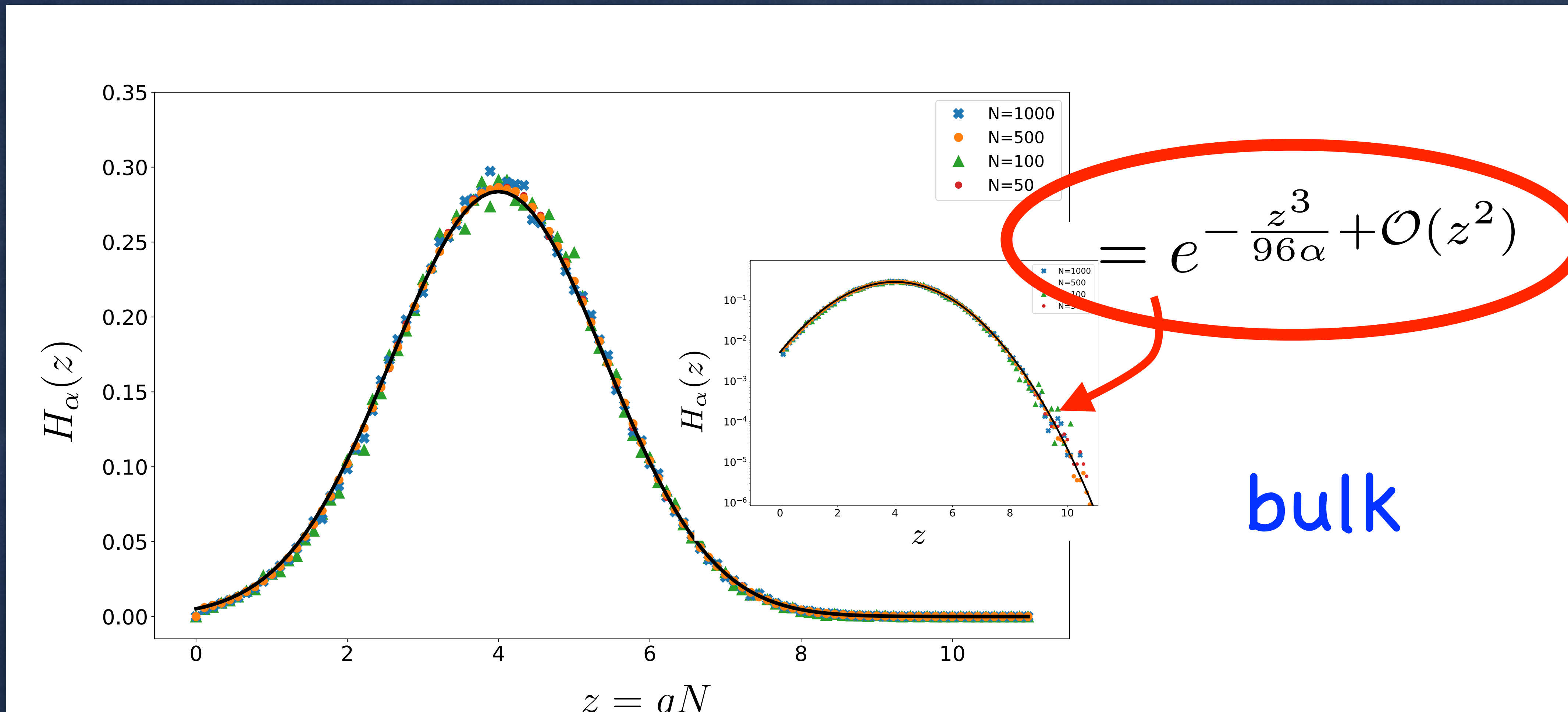
$$P_{\mathrm{gap,bulk}}(g,N) \sim \begin{cases} N\, H_\alpha(g\,N), & g \sim \mathcal{O}(1/N), \\ e^{-N^3 \psi_{\mathrm{bulk}}(g)}, & g \sim \mathcal{O}(1)\,, \end{cases}$$

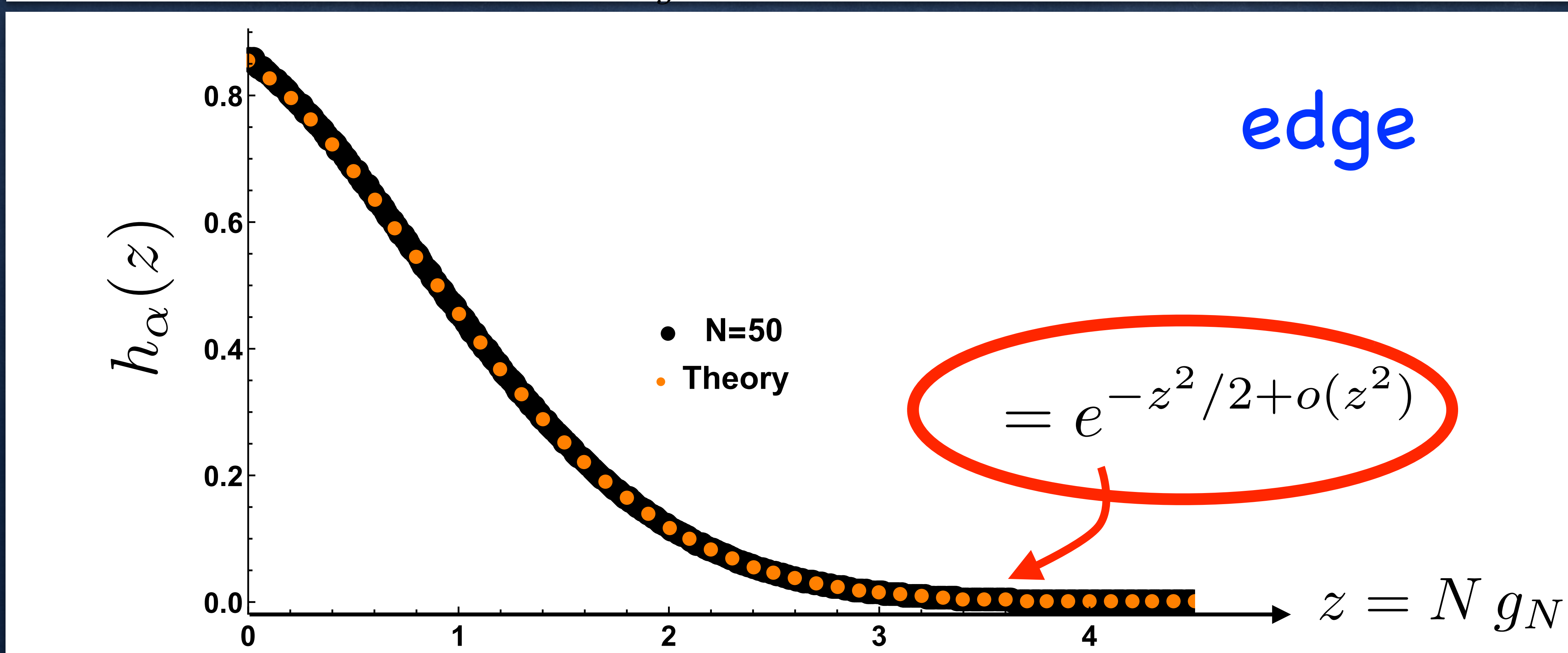
- Atypical/large fluctuations : $g = \mathcal{O}(1)$

$$\psi_{\mathrm{bulk}}(g) = \frac{g^3}{96\alpha}$$

i.e., the prolongation of the tail of $H_\alpha(z)$

Typical gap fluctuations: bulk vs edge





Conclusion and perspectives

- Exact extreme statistics of one-dimensional Coulomb gas
  - Typical fluctuations are NOT given by Tracy-Widom distributions
- Gap statistics: different behaviors in the bulk and at the edge
  - In the bulk, the distribution is NOT given by the Wigner surmise (i.e., corresponding to $N=2$ as in Gaussian β-ensembles) see also S. Santra et al. PRL 2022 for Riesz gas
  - Q: how to interpolate between these two regimes?
- Exact results for the full counting statistics (i.e. the number of particles in $[-L,+L]$) or for the index: strong hyperuniformity