Edge behavior in the 2d and 4d Quantum Hall effect

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Coulomb gases and universality, Jussieu 2022

[Estienne, Oblak & JMS, Scipost Physics, 2021]
[Cardoso, JMS & Abanov, Journal of Physics A, 2021]

Outline

Hamiltonian in a magnetic field + trapping potential

$$H = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + \frac{k}{2}(x^2 + y^2)$$

with

$$\mathbf{p} = -i\hbar \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \qquad \mathbf{A} = \frac{B}{2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

naturally leads to single particle wave functions

$$\phi_m(z) = \frac{z^m}{\sqrt{\pi m!}} e^{-|z^2|/2}$$

and ground state many-body wavefunction

$$\Psi(z_1, \ldots, z_N) = \det_{1 \le j,m \le N} \phi_{m-1}(z_j)$$

$$|\Psi(z_1, \ldots, z_N)|^2 = \prod_{j<k} |z_j - z_k|^\Gamma e^{-\sum_{j=1}^N |z_j|^2}$$

where $\Gamma = 2$. Other values of Γ: Laughlin state, log gas.

Well-known free fermions model, everything determined from the two-point function (or correlation kernel):

$$\begin{aligned} K_N(z, z') &= \sum_{m=0}^{N-1} \phi_m^*(z)\phi_m(z') \\ &= \sum_{m=0}^{N-1} \frac{(z^* z')^m}{\pi m!} e^{-(|z|^2+|z'|^2)/2} \end{aligned}$$

Density profile for large N

$K_N(z,z)$ is the density profile.

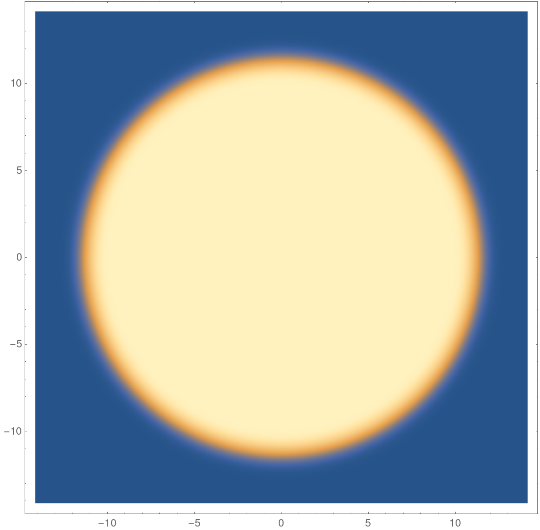
$$\lim_{N\to\infty} K_N(u\sqrt{N}, u\sqrt{N}) = \frac{1}{\pi}$$

for $u \in [0,1)$. Constant density in the bulk.

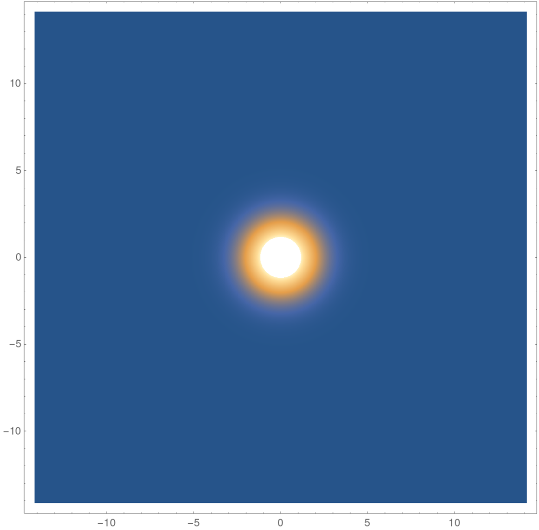
$$\lim_{N\to\infty} K_N(\sqrt{N}+x, \sqrt{N}+x) = \frac{\mathrm{erfc}(x\sqrt{2})}{2\pi}$$

Droplet is a disc with radius $\sqrt{N}$.

Density profile for large N



$|K(x_0, w)|$ for some x_0 on the real line



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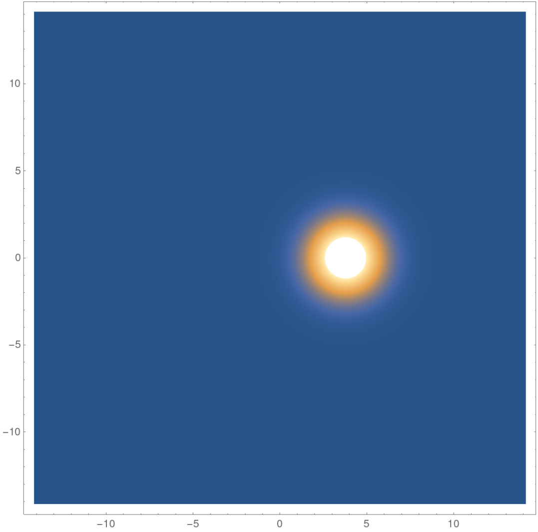

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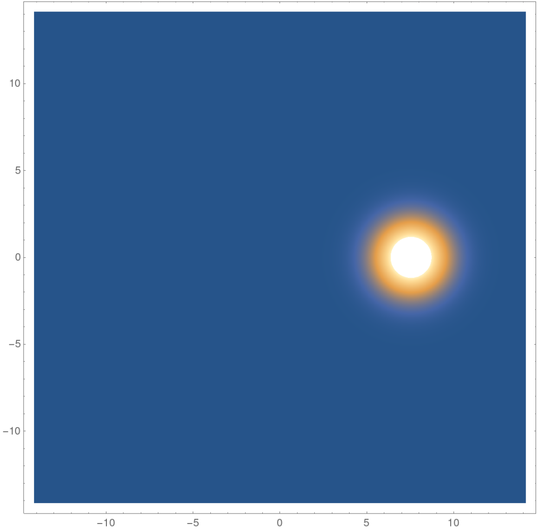

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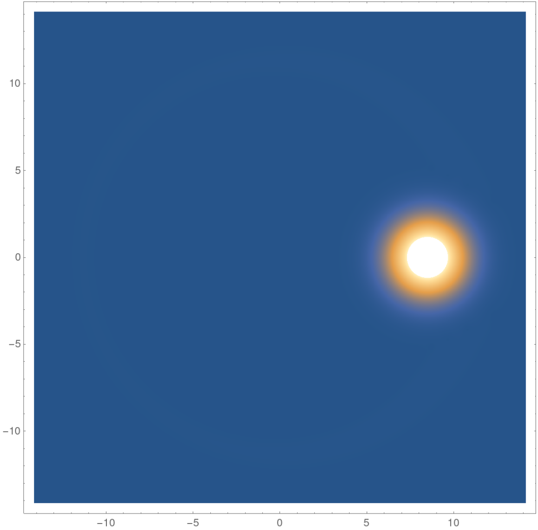

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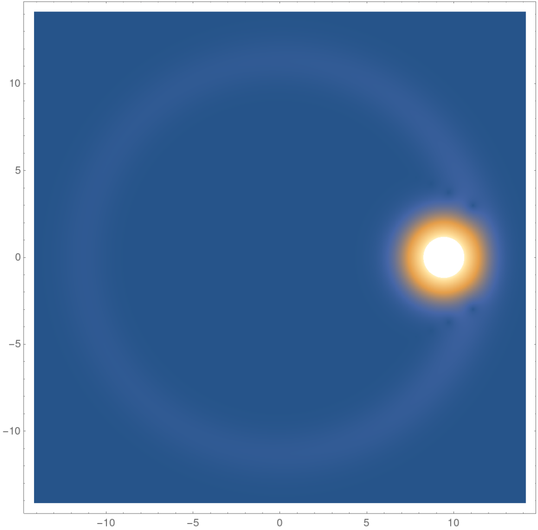

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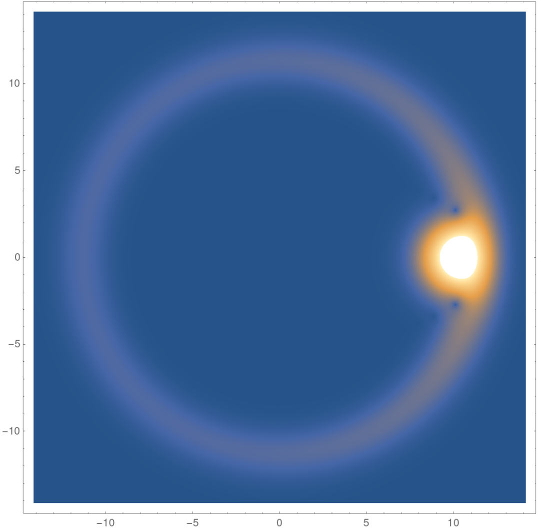

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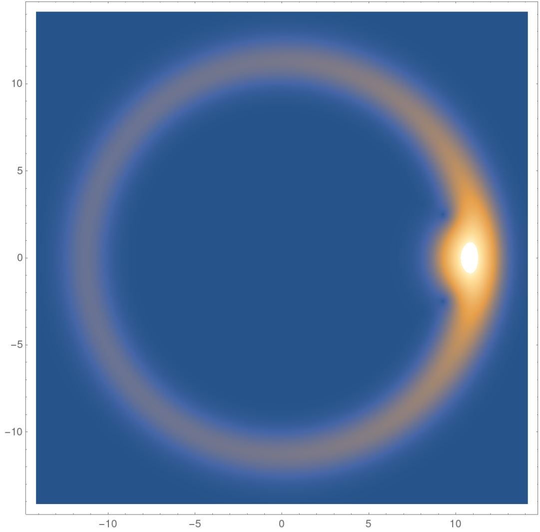

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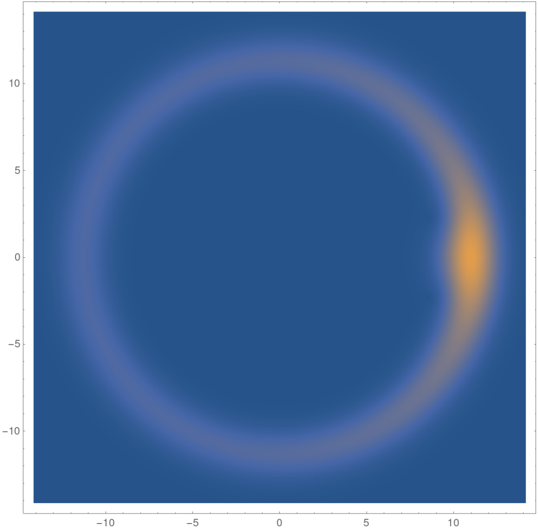

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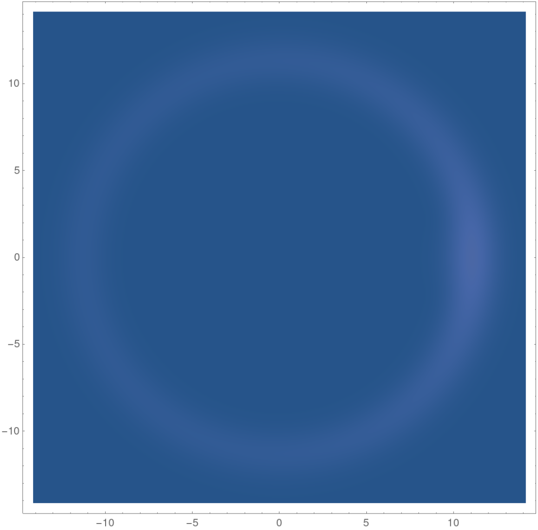


For large N and $\theta \in (0, 2\pi)$ [Jancovici 1982]:

$$|K_N(\sqrt{N}, \sqrt{N}e^{i\theta})| \sim \frac{\text{constant}}{\sqrt{N}\sin\frac{\theta}{2}}$$

Ideas of droplet deformations: Electrostatics and conformal maps [Choquard, Piller & Rentsch 1986], symmetries [Cappelli, Trugenberger & Zemba, 1993], generalizations [Zabrodin & Wiegmann 2006], free field, proofs [Leblé & Serfaty 2018], [Bauerschmidt, Bourgade, Nikula & Yau 2019], [Ameur & Cronvall 2022], conformal field theory [Read & Rezayi 1998].

Weird for now interpretation: $\sin\frac{\theta}{2} = \sin t$, where t is the time it takes to travel counterclockwise at constant speed on the unit circle, from $w = 1$ to $w = e^{i\theta}$ (“circumnavigation” takes time π).

Origin of this term

- $|\phi_m(z)|$ maximal for $|z| = \sqrt{m}$. Decays fast away from $\sqrt{m}$.
- Edge asymptotics for each single particle wavefunction :

$$\phi_{N-p}(\sqrt{N}e^{\mathrm{i}\theta}) \sim e^{\mathrm{i}N\theta}\, \frac{e^{-\mathrm{i}p\theta - p^2/(4N)}}{(2N\pi^3)^{1/4}}$$

- Finally perform the sum:

$$\sum_{p=1}^{N} \phi_{N-p}(\sqrt{N})\phi_{N-p}(\sqrt{N}e^{\mathrm{i}\theta})$$

4d Quantum Hall effect

$(x, y, u, v) \in \mathbb{R}^4$. Simplest generalization of the previous model:

$$H = (\mathbf{p} - \mathbf{A})^2 + \frac{k}{2}\left(x^2 + y^2\right) + \frac{k'}{2}\left(u^2 + v^2\right)$$

with vector potential

$$\mathbf{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \\ -v \\ u \end{pmatrix}$$

leads to ($z = x + iy$, $w = u + iv$ up to some units)

$$\phi_{m,n}(z, w) = \frac{z^m w^n}{\sqrt{\pi^2 m! n!}} e^{-(|z|^2 + |w|^2)/2}$$

with energies $\epsilon_{m,n} = \hbar(\omega m + \omega' n)$ with $\omega = \frac{k}{B}$, $\omega' = \frac{k'}{B}$

Generalizations of QHE to 4d predict anisotropic edge modes

[Zhang & Hu, Science 2001]

[Karabali & Nair, Nucl. Phys. B 2002]

[Helvang & Polschinski, CRP 2003]

Possible experimental realisations by engineering synthetic dimensions (quasicrystals, internal states, photonics,. . .)

[Kraus, Ringel & Zilberberg, PRL 2013]

[Price, Zilberberg, Ozawa, Carusotto & Goldman, PRL 2015]

[Lohse, Schweizer, Price, Zilberberg & Bloch Nature 2018]

[Bouhiron *et al.*, arXiv:2210.06322]

. . .

Single particle state (m, n) has energy $\varepsilon_{n,m} = \hbar(\omega n + \omega' n)$, where $\omega = k/B$, $\omega' = k'/B$.

Correlation kernel

$$K_N(z, w|z', w') = \sum_{m+\Delta n<N} \frac{(z^* z')^m (w^* w')^n}{\pi^2 m! n!} e^{-(|z|^2+|w|^2+|z'|^2+|w'^2|)/2}$$

$$\Delta = \frac{\omega'}{\omega}$$

Main results (I)

Droplet: squashed $4-$Ball $|z|^2 + \Delta|w|^2 \leq N$. Edge parametrization

$$z = \sqrt{N} e^{\mathrm{i}\alpha} \cos\frac{\theta}{2} \qquad , \qquad w = \sqrt{N} e^{\mathrm{i}\beta} \sin\frac{\theta}{2}$$

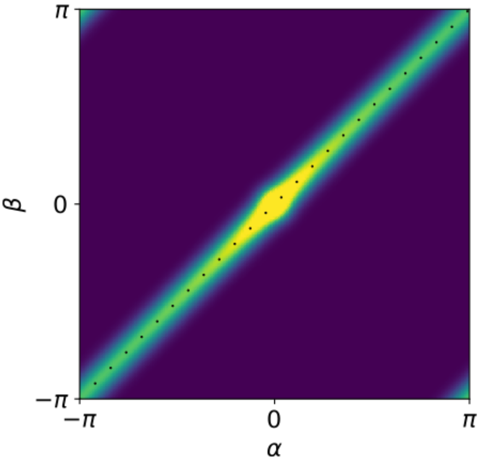
Without loss of generality, another point at the edge parametrized by $z' = \sqrt{N}\cos\frac{\theta'}{2}$, $w' = \sqrt{N}\sin\frac{\theta'}{2}$.

Can show with this parametrization that

$$\lim_{N\to\infty} \sqrt{N} K_N(z, w|z', w') = 0$$

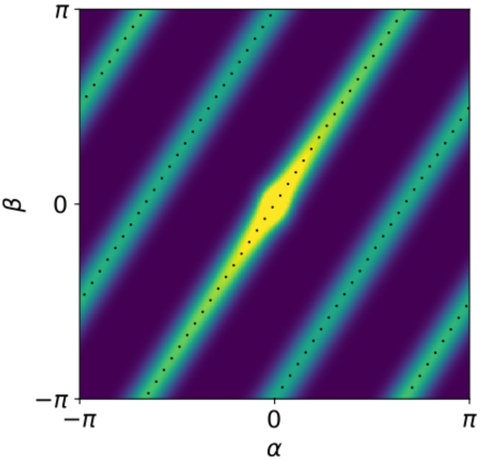
if $\theta \neq \theta'$, so edge modes necessarily live on the α, β torus.

Edge modes on the torus



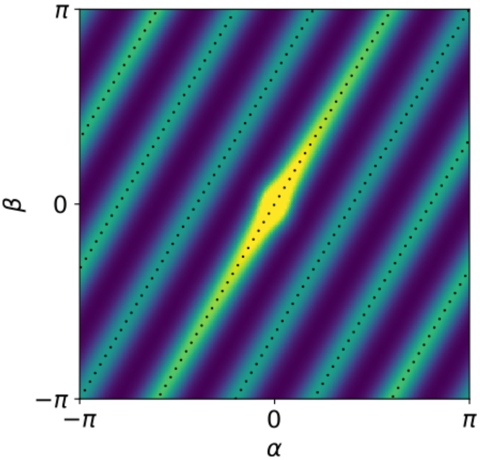
$\Delta = 1$

Edge modes on the torus



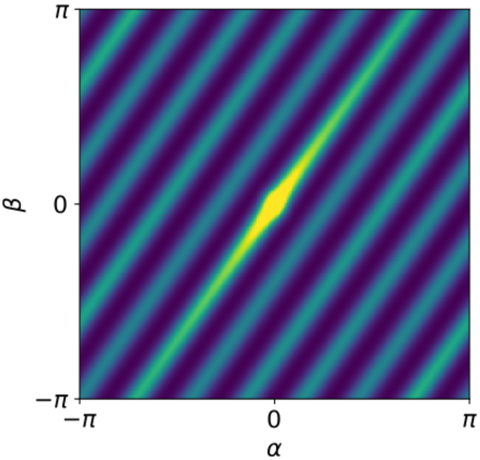
$$\Delta = 3/2$$

Edge modes on the torus



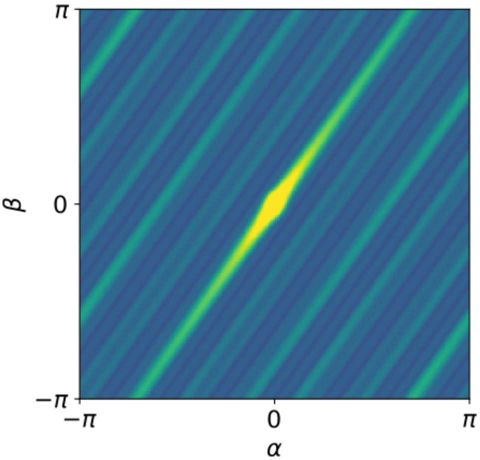
$\Delta = 5/3$

Edge modes on the torus



$$\Delta = 7/5$$

Edge modes on the torus



$$\Delta = 41/29$$

Main results (II)

Set $\theta = \theta'$ and define $\mathcal{K}(\alpha, \beta) = \lim_{N\to\infty} |\sqrt{N} K_N(z, w|z', w')|$

- $\mathcal{K}(\alpha, \beta) > 0$ iff $e^{\mathrm{i}\Delta\alpha} = e^{\mathrm{i}\beta}$.
- If $e^{\mathrm{i}\Delta\alpha} = e^{\mathrm{i}\beta}$ and $\Delta \in \mathbb{Q}$, then

$$\mathcal{K}(\alpha, \beta) = \frac{\mathrm{const}}{\sin t}$$

  Similar to before, t is the time it takes to reach (z, w) from (z', w') while traveling at constant speed on the torus.

- If $\Delta \notin \mathbb{Q}$

$$\mathcal{K}(\alpha, \beta) = \frac{\mathrm{const}'}{t}$$

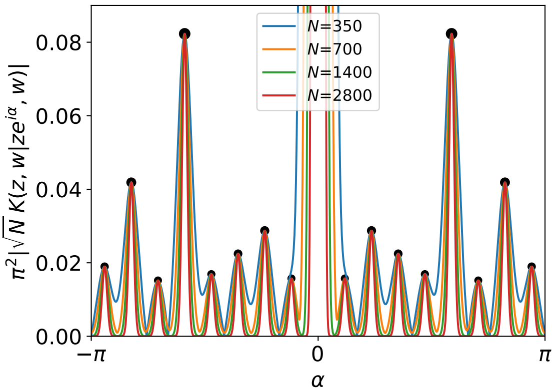
Main trick in the derivation

Rational case: same strategy as in $2d$. With $\Delta = p/q$, just use

$$\sum_{m+\Delta n<N} g(m,n) = \frac{1}{pq} \sum_{m'+n'<qN} \sum_{\mu=1}^{q} \sum_{\nu=1}^{p} e^{\mathrm{i}2\pi[\frac{\mu m'}{q} + \frac{\nu n'}{p}]} g(m'/q, n'/p)$$

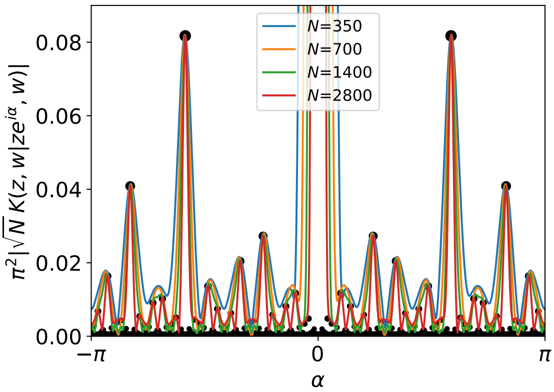
Irrational case: not so easy.

1d slices at $\beta = 0$



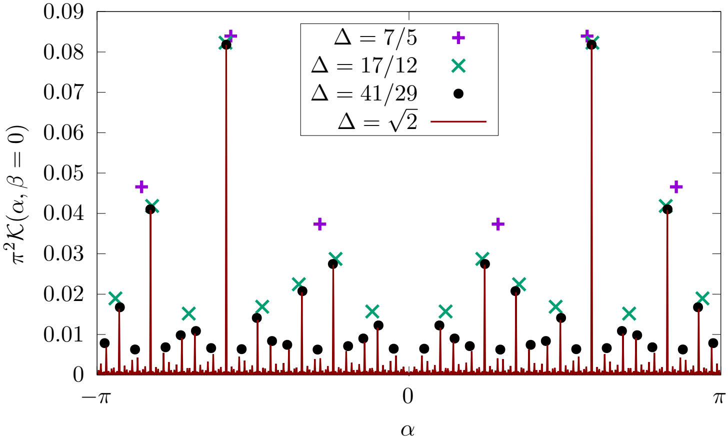
$\Delta = 17/12$

1d slices at $\beta = 0$



$$\Delta = \sqrt{2}$$

1d slices at $\beta = 0$



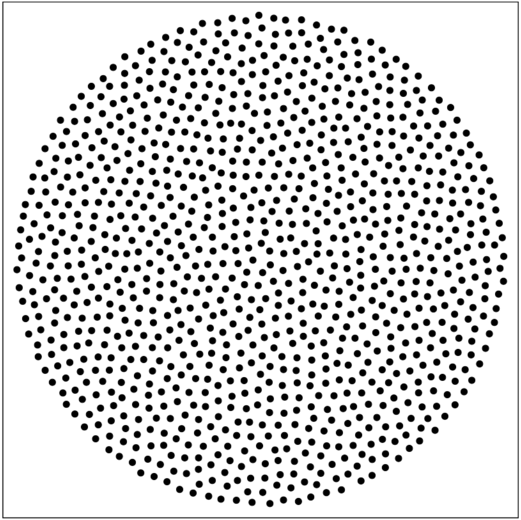
$\Delta = \sqrt{2}$

Edge density profile in the Laughlin state/2d log gas

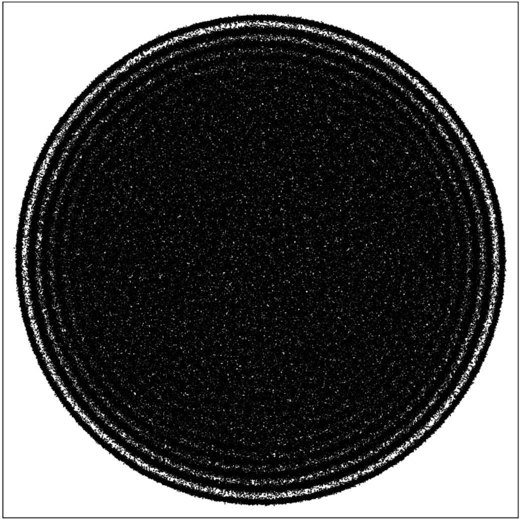
$$|\Psi(z_1,\ldots,z_N)|^2 = \frac{1}{Z_N(\Gamma)} \prod_{j<k} |z_j - z_k|^\Gamma e^{-\sum_{j=1}^N |z_j|^2}$$

for larger values of Γ.

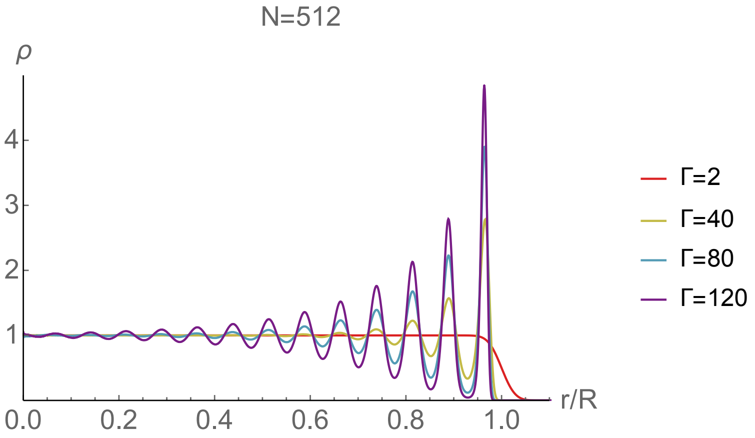
1024 particles, $\Gamma = 80$, 1 snapshots



1024 particles, $\Gamma = 80$, 500 superimposed snapshots



- Oscillations of the density close to the edge can be understood perturbatively around $\Gamma - 2$.
  [Jancovici 1981], [Can, Forrester, Tellez & Wiegmann 2014]
- Oscillations become larger as Γ is increased:



Assume crystalization to a triangular lattice for $\beta \to \infty$.

[Tkachenko 1966]

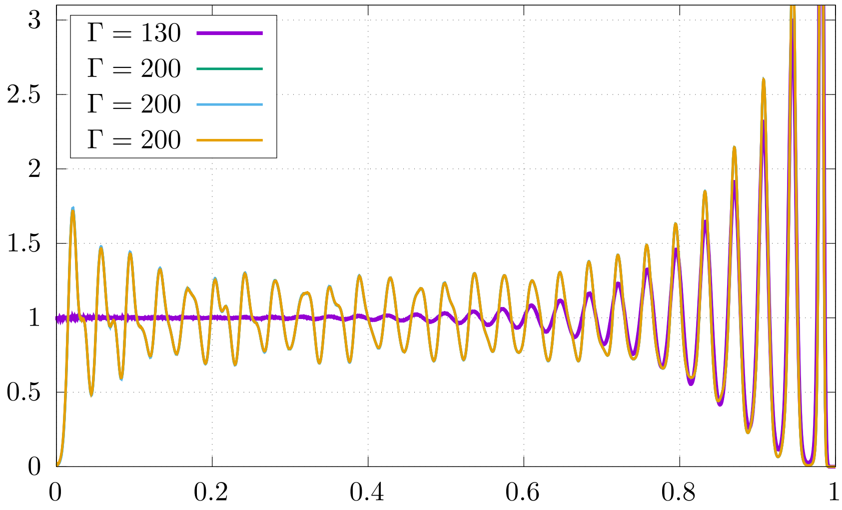
Phase transition observed in numerical simulations at $\Gamma \simeq 140$.

[Caillol, Levesque, Weis, Hansen 1981]

What we observe numerically:

- For $\Gamma < 140$ exponential decay of oscillations with distance to the edge. The peaks match the triangular lattice structure. Freezing at the edge, with crystal melted into the bulk.
- For $\Gamma > 140$ oscillations appear to decay much slower.

$N = 2048$ particles



The edge of a quantum Hall droplet is interesting.

Thank you!