Quantum Hall physics in higher Landau levels

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The quantum Hall effect(s)

The **integer quantum Hall effect** (IQHE) is a quantum phenomenon in strong magnetic fields. The Hall conductance of 2D electrons (fermions) shows precisely quantized plateaus when the filling factor (particle density/magnetic field strength) is an integer multiple of $\frac{e^2}{h}$. (von Klitzing 1980; Nobel prize 1985)

In the **fractional quantum Hall effect** (FQHE) the plateaus occur at fractional multiples of $\frac{e^2}{h}$ (Störmer, Tsu; Laughlin 1982-83; Nobel prizes1998, and 2016 to Haldane.)

A bosonic FQHE is theoretically possible in rapidly rotating cold atomic gases, or in synthetic gauge fields.

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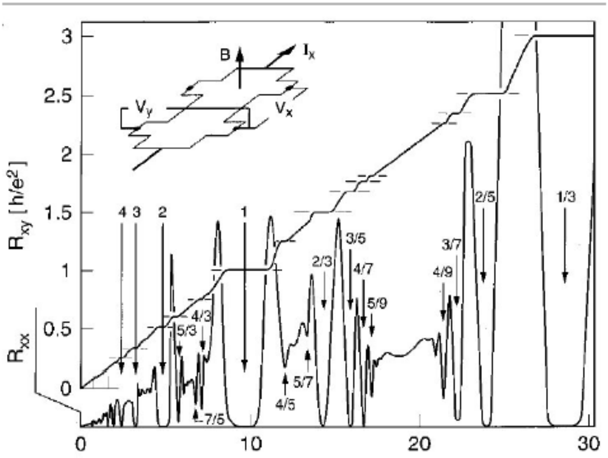
The FQHE. $R_{xy} = V_y/I_x; R_{xx} = V_x/I_x$



[Störmer-Tsui-Gossard *Rev. Mod. Phys.* 1999]

The basic Hamiltonian

Starting point: Hamiltonian for N (spinless, or spin polarized) fermions (electrons, charge $-e$) moving in $2\mathrm{D}$ with a perpendicular magnetic field:

$$H_N = \sum_{i=1}^{N} \left[H^{(i)}_{\mathrm{magn}} + V(\mathbf{r}_i) \right] + \sum_{i<j} w(\mathbf{r}_i - \mathbf{r}_j)$$

The external potential V models trapping and impurities. The interaction potential w is usually assumed to be repulsive Coulomb, i.e.,

$$w(\mathbf{r}_i - \mathbf{r}_j) = \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

The potential V includes then a background of opposite charge.

A strong magnetic field sets the dominant energy scale, $\sim B$.

The 1-particle magnetic Hamiltonian

The magnetic Hamiltonian is (in units so that $\hbar = 1$, $m = 1$)

$$H_{\rm magn} = \frac{1}{2}(\pi_x^2 + \pi_y^2)$$

where

$$\boldsymbol{\pi} = (\pi_x, \pi_y) = \mathbf{p} + e\mathbf{A}$$

is the gauge invariant kinetic momentum with

$$\mathbf{p} = -\mathrm{i}\hbar(\partial_x, \partial_y)$$

the canonical momentum and (in the symmetric gauge)

$$\mathbf{A} = \frac{B}{2}(-y, x).$$

The 1-particle magnetic Hamiltonian (cont.)

The kinetic momentum components satisfy the canonical commutation relations

$$[\pi_x, \pi_y] = \mathrm{i}\ell_B^{-2}$$

with the *magnetic length*

$$\ell_B = (eB)^{-1/2}.$$

In terms of the creation and annihilation operators

$$a^\dagger = \frac{\ell_B}{\sqrt{2}}(-\pi_y - \mathrm{i}\pi_x), \quad a = \frac{\ell_B}{\sqrt{2}}(-\pi_y + \mathrm{i}\pi_x)$$

with $[a, a^\dagger] = 1$ one can write

$$H_{\mathrm{magn}} = 2\ell_B^{-2}(a^\dagger a + \tfrac{1}{2}).$$

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The Landau levels

This is the Hamiltonian of a harmonic oscillator with eigenvalues

$$\epsilon_n = (n + \tfrac{1}{2})2eB, \ n = 0, 1, \ldots .$$

Every eigenvalue is infinitely degenerate. The degeneracy per unit area is $(2\pi\ell_B^2)^{-1} \sim B$.

The eigenspace with $n = 0$ is called the lowest Landau level, denoted LLL. The n-the Landau level is denoted $n\mathrm{LL}$.

The corresponding fermionic spaces for N electrons are denoted $\mathrm{LLL}^{\otimes_a N}$ and $n\mathrm{LL}^{\otimes_a N}$ respectively.

The degeneracy can be parametrized by the angular momentum operator $\mathbf{r} \times \mathbf{p}$, or equivalently, by the eigenvalues of another harmonic oscillator H'_{magn}, commuting with H_{magn}, and associated with guiding centers.

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Cyclotron motion and guiding centers

The Landau spectrum arises through the quantization of the cyclotron motion around guiding centers. One arrives at this picture by writing the gauge invariant position operator $\mathbf{r}$ as

$$\mathbf{r} = \mathbf{R} + \widetilde{\mathbf{R}}$$

with the guiding center part $\mathbf{R}$ and the Landau orbit (cyclotron) part

$$\widetilde{\mathbf{R}} = \ell_B^2 \mathbf{n} \times \boldsymbol{\pi},$$

where $\mathbf{n}$ the unit normal vector to the plane.

Both $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ are gauge invariant and and $[\mathbf{R}, \widetilde{\mathbf{R}}] = \mathbf{0}$ while

$$[R_x, R_y] = -\mathrm{i}\ell_B^2, \quad [\widetilde{R}_x, \widetilde{R}_y] = \mathrm{i}\ell_B^2.$$

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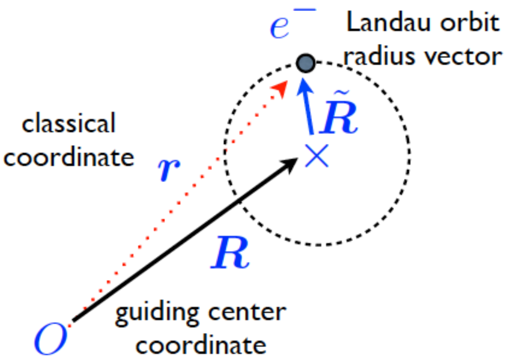

The two oscillators

The creation and annihilation operators for the cyclotron oscillator $H_{\rm magn}$ are in terms of $\widetilde{\mathbf{R}}$

$$a^\dagger = \frac{1}{\sqrt{2}\ell_B}(\widetilde{R}_x - \mathrm{i}\widetilde{R}_y), \quad a = \frac{1}{\sqrt{2}\ell_B}(\widetilde{R}_x + \mathrm{i}\widetilde{R}_y).$$

The Hamiltonian $H'_{\rm magn}$ for the guiding centers on the other hand is

$$H'_{\rm magn} = 2\ell_B^{-2}(b^\dagger b + \tfrac{1}{2})$$

with

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Complex notation

Choose units so that $eB = 2$, i.e., $\ell_B = 1/\sqrt{2}$.

The two-dimensional configuration space $\mathbb{R}^2$ can be identified with the complex plane $\mathbb{C}$.

Defining complex coordinates and derivatives by

$$z = x + \mathrm{i}y, \quad \bar{z} = x - \mathrm{i}y, \quad \partial_z = \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y), \quad \partial_{\bar{z}} = \tfrac{1}{2}(\partial_x + \mathrm{i}\partial_y)$$

we can write

$$a^\dagger = \tfrac{1}{2}\bar{z} - \partial_z, \quad a = \tfrac{1}{2}z + \partial_{\bar{z}}, \qquad b^\dagger = \tfrac{1}{2}z - \partial_{\bar{z}}, \quad b = \tfrac{1}{2}\bar{z} + \partial_z$$

Recall that the $a^\#$ are associated with the cyclotron oscillator $\widetilde{\mathbf{R}}$ and raise or lower the LL index, while the $b^\#$ are associated with the guiding center oscillator $\mathbf{R}$ and leave each LL fixed.

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Eigenfuntions

The two sets of creation and annihilation operator generate the whole 1-particle Hilbert space $L^2(\mathbb{C}, \mathrm{d}^2 z)$ The gaussian wave function

$$\varphi_{0,0}(z, \bar{z}) := \frac{1}{\sqrt{\pi}} e^{-|z|^2/2}$$

is the common ground state for the commuting harmonic oscillators

$$H = 4\left(a^\dagger a + \tfrac{1}{2}\right) \quad \text{and} \quad H' := 4\left(b^\dagger b + \tfrac{1}{2}\right).$$

The states

$$\varphi_{n,m} = (a^*)^n (b^*)^m \varphi_{0,0}$$

form a basis of common eigenfunctions with

$$H\varphi_{n,m} = 4\left(n + \tfrac{1}{2}\right)\varphi_{n,m}, \quad H'\varphi_{n,m} = 4\left(m + \tfrac{1}{2}\right)\varphi_{n,m}.$$

Holomorphy

The functions $\varphi_{n,m}$ with n fixed, $m = 0, 1, \ldots$ span the nth-Landau level $n\mathrm{LL}$. The lowest Landau level LLL is spanned by the $\varphi_{0,m} \sim z^m e^{-|z|^2/2}$; its wave functions have the form

$$\psi(z, \bar{z}) = f(z)e^{-|z|^2/2}$$

with holomorphic f.

Functions in $n\mathrm{LL}$ have the form

$$\psi(z, \bar{z}) = \sum_{k=0}^{n} f_k(z)\bar{z}^k e^{-|z|^2/2}$$

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The Laughlin wave function

So far, only "free" 1-particle states have been discussed. The FQHE is, however, a many body phenomenon and interactions are essential.

The Laughlin wave function (Laughlin 1983), is an ansatz for the ground state of the many-body Hamiltonian in the N-particle lowest Landau level $\mathrm{LLL}^{\otimes_a N}$ to explain filling fractions $1/q$ in the FQHE, $q \geq 3$ odd and small. It is defined by

$$\Psi_{\rm Lau}^{(q)} = C_{N,\ell} \prod_{i<j} (z_i - z_j)^q e^{-\sum_{i=1}^N |z_i|^2/2}$$

Its "quasi-hole", $\prod_i (z_i - \eta)$ and "quasi-particle", $\prod_i (\partial_{z_i} - \bar{\eta})$ excitations exhibit fractional charges $\pm 1/q$ and fractional statistics.

By estimating the energies of these excitations Laughlin concluded that for $q \leq 7$ his wave function describes an **incompressible fluid**.

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Plasma analogy

Important properties **can**, however, be rigorously proved using Laughlin's plasma analogy. This is the interpretation of the N-particle probablilty density $|\Psi^{(q)}_{\rm Lau}|^2$ as a Boltzmann-Gibbs factor for a 2D classical Coulomb gas in a neutralizing background:

$$N^N|\Psi^{(q)}_{\rm Lau}(\sqrt{N}\vec{z})|^2 = \mathcal{Z}_N{}^{-1}\exp\left(-\beta\mathcal{H}_N(\vec{z})\right)$$

with $\beta = N$ and the classical Hamiltonian function

$$\mathcal{H}_N(\vec{z}) = \sum_{j=1}^{N}|z_j|^2 + \frac{2q}{N}\sum_{1\leq i<j\leq N}\log\frac{1}{|z_i - z_j|}.$$

Using this representation Laughlin concluded that for large N his wave function describes a droplet of radius $\sqrt{qN}$ and uniform density $(q\pi)^{-1}$, except close to the edges. (Rigorous proof by Rougerie, Serfaty and JY 2014.)

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Density bounds, rigidity

More generally we may consider perturbations of $|\Psi_{\rm Lau}^{(q)}|^2$ of the kind

$$\Psi_F = \Psi_{\rm Lau}^{(q)} F(\vec{z})$$

with a holomorphic function F. This amounts to adding a plurisuperharmonic function

$$\mathcal{W}_N(\vec{z}) := -\frac{2}{N} \log \left| F\left(\sqrt{N}\, \vec{z}\right)\right|.$$

to $\mathcal{H}_N$.

In 2018 Lieb, Rougerie and JY proved that the particle density of Ψ_F (suitably averaged) cannot exceed Laughlin's bound $(\pi q)^{-1}$.

This important fact was called rigidity of the Laughlin state by LRY.

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The unitary correspondence nLL $\leftrightarrow$ LLL

We now come to the **main message** of this lecture:

All results based on holomorphy it in the LLL have counterparts in higher LL's. In particular there are Laughlin states in all LL's. The reason is that **all LL's are unitarily equivalent in a natural way!**

Consequently can always work with holomorphic wave functions in the LLL, but the point to note is that particle densities, or equivalently, interaction potentials may depend on the LL. For this reason a state can be compressible in one LL while its counterpart in another LL is incompressible.

Different roads to the correspondence $n\mathrm{LL} \leftrightarrow \mathrm{LLL}$:

- Coherent states for the guiding center oscillator
- The formula $\exp(i\mathbf{q}\cdot\mathbf{r}) = \exp(i\mathbf{q}\cdot\mathbf{R})\exp(i\mathbf{q}\cdot\tilde{\mathbf{R}})$
- Use the creation and annihilation operators $(a^*)^n$ and a^n.

Each of the three roads sheds a different light on the correspondence, but the last one is the simplest from a mathematical point of view.

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Different roads to the correspondence $n\mathrm{LL} \leftrightarrow \mathrm{LLL}$:

- Coherent states for the guiding center oscillator
- The formula $\exp(\mathrm{i}\mathbf{q}\cdot\mathbf{r}) = \exp(\mathrm{i}\mathbf{q}\cdot\mathbf{R})\exp(\mathrm{i}\mathbf{q}\cdot\tilde{\mathbf{R}})$
- Use the creation and annihilation operators $(a^*)^n$ and a^n.

Each of the three roads sheds a different light on the correspondence, but the last one is the simplest from a mathematical point of view.

The unitary correspondence $n\mathrm{LL} \leftrightarrow \mathrm{LLL}$

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A direct approach

The last mentioned approach is based on the unitary operators $U_n : n\mathrm{LL} \to \mathrm{LLL}$ and their inverses $U_n^{-1} : \mathrm{LLL} \to n\mathrm{LL}$ defined by

$$U_n = (n!)^{-1/2} a^n \restriction n\mathrm{LL}, \qquad U_n^{-1} = (n!)^{-1/2} (a^\dagger)^n \restriction \mathrm{LLL}.$$

Using the formulas $a = \frac{1}{2}z + \partial_{\bar z}$, $a^\dagger = \frac{1}{2}\bar z - \partial_z$, it follows that

If $\psi_n \in n\mathrm{LL}$ has wave function $\psi_n(z, \bar z) = \sum_{k=0}^n f_k(z) \bar z^k e^{-|z|^2/2}$, $f_0, \dots, f_n$ holomorphic, then the wave function of $\psi_0 = U_n \psi_n \in \mathrm{LLL}$ is

$$\psi_0(z, \bar z) = \sqrt{n!} f_n(z) e^{-|z|^2/2},$$

with

$$f_k(z) = (-1)^{n-k} \binom{n}{k} \partial_z^{n-k} f_n(z).$$

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Comparison of ℓ-particle densities in different LL's

The main result on ℓ-particle densities in higher Landau levels, obtained by means of the formula

$$U_n = (n!)^{-1/2} a^n = (n!)^{-1/2} (\tfrac{1}{2} z + \partial_{\bar{z}})^n$$

and reshuffling of differentations, is:

Theorem

The ℓ-particle densities of $\Psi_n \in n\mathrm{LL}^{\otimes_a N}$ and $\Psi_0 = U_n^{\otimes N} \Psi_n \in \mathrm{LLL}^{\otimes_a N}$ are connected by

$$\rho^{(\ell)}_{\Psi_n}(\mathbf{r}_1, \dots, \mathbf{r}_\ell) = \prod_{i=1}^{\ell} L_n\left(-\tfrac{1}{4}\Delta_{\mathbf{r}_i}\right) \rho^{(\ell)}_{\Psi_0}(\mathbf{r}_1, \dots, \mathbf{r}_\ell)$$

where $L_n(t) = \sum_{l=0}^{n} \binom{n}{l} \frac{(-t)^l}{l!}$ is the n-th Laguerre polynomial.

Corollaries

1. The local density of a perturbed Laughlin state in $n\mathrm{LL}^{\otimes N}$,

$$\Psi_n = F(b_1^\dagger, \dots, b_N^\dagger) \prod_{i<j} (b_i^\dagger - b_j^\dagger)^q \varphi_{n,0}^{\otimes N}$$

with holomorphic F satisfies for $N \to \infty$ the density upper bound $(q\pi)^{-1}$ independently of n, when averaged by suitably regularized characteristic functions on scales $\ll \sqrt{N}$.

(In addition full, “inert” lower levels contribute $1/\pi$ to the total density each.)

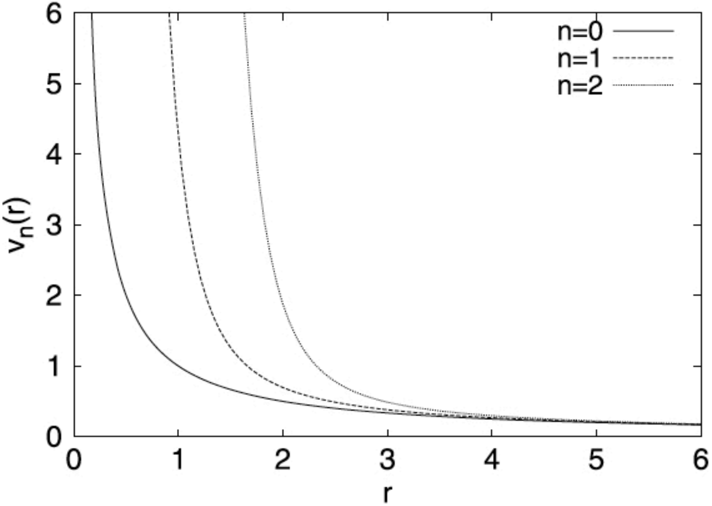
2. Potential energies of a state in $n\mathrm{LL}^{\otimes_a N}$ can be evaluated using the corresponding wave function in $\mathrm{LLL}^{\otimes_a N}$ if the external potential $V(\mathbf{r})$ is replaced by the effective external potential

$$V(\mathbf{r})_n^{\mathrm{eff}} = L_n\left(-\tfrac{1}{4}\Delta_{\mathbf{r}}\right) V(\mathbf{r})$$

and the interaction potential $w(\mathbf{r}_1, \mathbf{r}_2)$ by the effective interaction potential

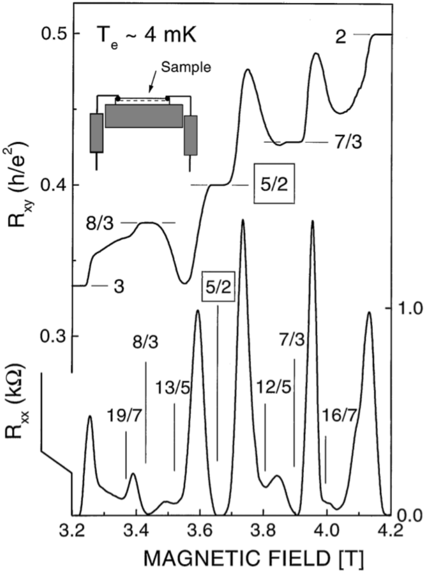
$$w(\mathbf{r}_1, \mathbf{r}_2)_n^{\mathrm{eff}} = L_n\left(-\tfrac{1}{4}\Delta_{\mathbf{r}_1}\right) L_n\left(-\tfrac{1}{4}\Delta_{\mathbf{r}_2}\right) w(\mathbf{r}_1, \mathbf{r}_2).$$

Effective Coulomb



The $\nu = 5/2$ FQHE

A Quantum Hall state with a clear signature for filling factor $\nu = 5/2$ was first observed in 1987. It arises from a half-filled LL with $n = 1$ on top of two full LLL's ($n = 0$) with opposite spin. This state is unusual since the great majority of observed plateaus in the Quantum Hall conductance correspond to filling factors with an odd denominator. Moreover, there is no sign of a plateau in the Hall resistance at $\nu = 1/2$ for $n = 0$ alone.



T_e ~ 4 mK
Sample
2
7/3
5/2
8/3
3
5/2
1.0
0.5
0.4
0.3
8/3
7/3
13/5
12/5
19/7
16/7
0.0
3.2
3.4
3.6
3.8
4.0
4.2
MAGNETIC FIELD [T]
R_xy (h/e^2)
R_xx (kΩ)

The $\nu = 5/2$ FQHE (cont.)

The theoretical explanation of the $\nu = 5/2$ FQHE is still a debated subject but the transformation formula between Landau levels for the Coulomb interaction should play an important role. Possibly the simple picture based on the mapping $n\mathrm{LL} \leftrightarrow \mathrm{LLL}$ is not enough, however, due to Landau level mixing.

Anyway, the observed $\nu = 5/2$ state is theoretically very interesting because proposed candidates for its wave function (Moore-Read Pfaffian states and variants) offer the possibilty of nonabelian braid statistics for its quasi-hole excitations.

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Pfaffian states

A Moore-Read state in the LLL for an even particle number N has the form

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{N}(z_i - z_j)^q \prod_{i=1}^{N} e^{-|z_i|^2/2}$$

with the Pfaffian

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathcal{A}\left\{\frac{1}{z_1 - z_2}\cdots\frac{1}{z_{N-1} - z_N}\right\}.$$

Here $\mathcal{A}$ stands for antisymmetrization over all possible pairings of the coordinates. It has filling fraction $1/q$ and is antisymmetric for *even* values of q. It is also holomorphic because singular factors in the Pfaffian are compensated by the factors $(z_i - z_j)^q$.

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Conclusions

- QH states for many-body systems in arbitrary Landau levels can be mapped unitarily onto states in the lowest Landau level.
- The basis for this is the splitting of the classical (commutative) position variables $\mathbf{r}$ into guiding centers and cyclotron variables.
- The motion of the guiding centers is independent of the Landau level. It is, however, decorated by the cyclotron motion which depends on the level and modifies particle densities expressed in the position variables.
- Particle densities and interactions in higher Landau levels can be studied in terms of suitably transformed densities and interactions in the lowest Landau level.
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