Coulomb gases and universality

Organized* by Mathieu Lewin & Sylvia Serfaty

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- **Angel Alastuey (École Normale Supérieure de Lyon)**

Breakdown of exponential screening at the critical point of a classical electrolyte

- **Guillaume Barraquand (École Normale Supérieure)**

Non-intersecting directed polymers, random surfaces and a disordered Coulomb gas

Many probabilistic models of non-intersecting paths can be seen as two-dimensional random surfaces, which, typically, converge to the Gaussian free field at large scale. In this talk, we will discuss the effect of a random environment. We will consider a Dyson Brownian motion in quenched disorder, that is, equivalently, a system of N non-intersecting directed polymers in $1+1$ dimensions. Our main result is a simple and explicit description of the stationary measure for this system (as well as for some discrete integrable variant). In particular, in presence of a random environment, the one dimensional Coulomb gas, i.e. the GUE density, becomes an explicit exponential functional of N Brownian motions. We will show some of its basic properties and discuss some conjectures. Based on joint work with Pierre le Doussal.

- **Gaëtan Borot (Berlin)**

Asymptotics in 1d log-like gases and open problems

I will review various aspects of the large N asymptotics in the macroscopic regime in systems of N particles on the real line having 2d Coulomb repulsion at short distance: equilibrium measure and how to compute it, large deviations, loop equations, how to extract from it all-order asymptotic expansions in the one-cut and the multi-cut regime. This will be illustrated with various models of physical or mathematical interest and some open problems. This will also serve as an introduction to some aspects discussed in the talk of Alice Guionnet.

- **Jeanne Boursier (École Normale Supérieure de Lyon)**

Decay of correlations for the circular Riesz gas

In this talk I will present results concerning the decay of correlations for the circular long-range Riesz gas, a system of particles interacting pairwise through an inverse power kernel. Our method is based on the analysis of the so-called Heffer-Sjöstrand equation or Witten Laplacian which gives a representation of correlations. I will explain how elementary analytical tools lead to optimal decay estimates on solutions. As a byproduct we prove the convergence of the microscopic process to a unique infinite volume process $\mathrm{Riesz}_{s,\beta}$, believed to be widely universal.

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• **Djalil Chafai (Paris Dauphine & École Normale Supérieure)**

Riesz energy problems and integral identities: Unexpected phenomena for equilibrium measures

I will present recent explorations in collaboration with Edward B. Saff and Robert S. Womersley, regarding the equilibrium measure in dimension d associated to a Riesz s-kernel interaction with an external field given by a power of the Euclidean norm. Our study reveals that the equilibrium measure can be a mixture of a continuous part and a singular part. Depending on the value of the power, a threshold phenomenon occurs and consists of a dimension reduction or condensation on the singular part. In particular, in the logarithmic case $s = 0$ ($d = 4$), there is condensation on a sphere of special radius when the power of the external field becomes quadratic. This contrasts with the case $d = s + 3$ which shows that the equilibrium measure is fully dimensional and supported on a ball. A lot is still to be explored for a global picture over all values of the parameters. Our approach makes use, among other tools, of the Frostman or Euler-Lagrange variational characterization, the Funk-Hecke formula, the Gegenbauer orthogonal polynomials, hypergeometric special functions, and elliptic integrals.

• **David Dereudre (Lille)**

Number-Rigidity and β-Circular Riesz gas

For an inverse temperature $\beta > 0$, we define the β–circular Riesz gas on $\mathbb{R}^d$ as any microscopic thermodynamic limit of Gibbs particle systems on the torus interacting via the Riesz potential $g(x) = \|x\|^{-s}$. We focus on the non integrable case $d - 1 < s < d$. Our main result ensures, for any dimension $d \geq 1$ and inverse temperature $\beta > 0$, the existence of a β–circular Riesz gas which is not number-rigid. Recall that a point process is said number rigid if the number of points in a bounded Borel set Δ is a function of the point configuration outside Δ. It is the first time that the non number-rigidity is proved for a Gibbs point process interacting via a non integrable potential. We follow a statistical physics approach based on the canonical DLR equations. It is inspired by Dereudre-Hardy-Leblé-Maïda (2021) where the authors prove the number-rigidity of the Sine–β process.

• **Laure Dumaz (École Normale Supérieure)**

Some aspects of the Anderson Hamiltonian with white noise

In this talk, I will present several results on the Anderson Hamiltonian with white noise potential in dimension 1. This operator formally writes "−Laplacian + white noise". It arises as the scaling limit of various discrete models and its explicit potential allows for a detailed description of its spectrum. We will discuss localisation of its eigenfunctions and local statistics of its eigenvalues. Around large energies, we will see that the eigenfunctions are localized and follow a universal shape given bu the exponential of a Brownian motion plus a drift, a behavior already observed by Rifkind and Virag in tridiagonal matrix models. Based on joint works with Cyril Labbé.

• **Benoît Estienne (Sorbonne)**

Cornering the universal shape of fluctuations

Understanding the fluctuations of observables is one of the main goals in physics. We investigate such fluctuations within a subregion of the full system, focusing on the dependence on the shape of the region. In particular for two-dimensional systems, corner contributions are universal: up to a numerical prefactor, the dependence on the opening angle does not depend on anything, provided the system under study is uniform, isotropic, and correlations do not decay too slowly. The prefactor contains important physical information: we show in particular that it gives access to the long-wavelength limit of the structure factor. We illustrate our findings with several examples: classical fluids, fractional quantum Hall (FQH) states, and scale invariant quantum critical theories.

- **Bertrand Eynard (CEA)**

A short review on matrix integrals vs Dyson gaz. Path homology, and large N expansions

The Lebesgue measure on the set of Hermitian (resp. Normal) $N \times N$ matrices, induces a measure on the set of eigenvalues, which takes the form of a Dyson gaz. This observation, going back to the founders of Random Matrix Theory, Dyson, Mehta, Gaudin..., gave rise to the method of "orthogonal polynomials". Years later, random matrices found another application in combinatorics of maps, and there, the large N expansion was easy to compute by combinatorics methods. The Schwinger-Dyson equations (corresponding to a recursion on the number of edges of graphs in the combinatorics of maps) are easy to solve recursively as formal power series of $1/N$. The solution of this recursive procedure takes a universal form, known as "topological recursion", and enjoy many beautiful algebraic geometry properties. Proving that the formal series solution coincides with the actual asymptotic expansion is often a hard task, and has been completed in a rather large range of cases, and remains conjectural for others. However, it gives a deep heuristical insight. Moreover, random matrices are known to be "integrable systems", which implies integrability properties for the Dyson gaz. This is related to the fact that many asymptotic limits give rise to Fredholm determinants and Tau functions of some KdV hierarchies.

- **Yan Fyodorov (King's college London)**

"Escaping the crowds": extreme values and outliers in rank-1 non-normal deformations of GUE/CUE

Rank-1 non-normal deformations of standard random matrices (Hermitian like GUE or unitary like CUE) provide the simplest model for describing the scattering matrix poles (aka "resonances") in a quantum chaotic system decaying via a single open channel. The joint probability density of those poles provide examples of a 2D Coloumb gas in the complex plane (in the half-plane for deformed GUE or the unit disk for CUE) subject to certain global constraints. Of particular interest are imaginary parts for deformed GUE or distance to the unit circe for CUE, identified with the resonance "widths". In the case of deformed GUE we provide a detailed description of an abrupt restructuring of the resonance density in the complex plane as the function of deformation parameter, identify the critical scaling of typical extreme imaginary parts, and finally describe how an atypically "broad" resonance (an outlier) emerges from the crowd. In the case of deformed CUE we are further able to study the Extreme Value Statistics of the "widest resonance" most remote from the unit circle. We find that in the critical regime it is described by a distribution nontrivially interpolating between Gumbel and Frechet. The presentation will be based on the joint works with Boris Khoruzhenko and Mihail Poplavskyi.

• **Christophe Garban (Lyon)**

Debye screening and GFF fluctuations for the Coulomb gas on $\mathbb{Z}^d$

The goal of this talk will be to present some puzzling properties of the (two-component) lattice Coulomb gas on the d-dimensional lattice. Some of these pro– perties will be "classical" (such as the Debye screening), some others will be based on joint works with Avelio Sepúlveda. The connection of this model with integer-valued fields and compact-valued spin systems will be emphasized through the talk.

• **David Garcia-Zelada (Sorbonne)**

Determinantal Coulomb gases in an uncharged region

When n classical electrostatic particles are confined by a fixed background of opposite charge, they end up canceling part of it as n grows to infinity. Nevertheless, there may be regions where this cancellation allows part of the particles to no longer feel this background charge. In other words, they end up interacting only between themselves while being confined to this region. In this occasion, I will present a description of these remaining particles or "outliers" in the two-dimensional determinantal setting. It will be seen that there is a unique behavior if the region has no holes but that the space of possible behaviors increases in dimension with the number of holes. This is joint work with Raphaël Butez, Alon Nishry and Aron Wennman (arXiv:2104.03959).

• **Alice Guionnet (École Normale Supérieure de Lyon)**

Matrix models and Dyson-Schwinger equations

In this talk I will discuss the uses of Dyson-Schwinger equations to analyze matrix models, and in particular Coulomb gases. At the end of the talk I will focus on recent work with E. Maurel Segala about low temperature multi-matrix models.

• **Markus Holzmann (Grenoble)**

Ground state phase diagram of jellium: what do we think we know?

The homogeneous electron gas (jellium) where electrons interact with each other and with a positive background charge is one of the simplest model system in condensed matter. Still, the precise determination of the zero temperature phase diagram remains challenging. In the talk I will review some recent progress from a computational perspective. I will first address ground state calculations within the Hartree-Fock approximation. Although Overhauser proved an instability of the free Fermi gas under spin- or charge density waves at high densities [1], explicit Hartree-Fock solutions have only been established half a decade later [1–4]. At low densities, the exchange energy favors a transition from a spin unpolarized to a fully polarized state, but already Wigner pointed out the importance of correlations beyond Hartree-Fock for this ferromagnetic transition. Introducing a repulsive energy term between opposite spin electrons to phenomenologically account for correlation effects, Stoner predicted the occurrence of a continuous magnetization transition. Based on most recent quantum Monte Carlo caculations [5], I will argue that a possible ferromagnetic transition at low densities is preculded by the formation of the Wigner crystal. Proper inclusion of correlations quite generally results in a suppression of Stoner's instability in all known ground states of homoge– neous quantum fluids with spin-independent repulsive interactions [5,6,7].

[1] A. W. Overhauser, Phys. Rev. Lett. 4, 462 (1960); Phys. Rev. 128, 1437 (1962)
[2] L. Baguet, F. Delyon, B. Bernu, and M. Holzmann, Phys. Rev. Lett. 111, 166402 (2013)
[3] F. Delyon, B. Bernu, L. Baguet, and M. Holzmann, Phys. Rev. B 92, 235124 (2015)
[4] D. Gontier, C. Hainzl, and M. Lewin, Phys. Rev. A 99, 052501 (2019)

[5] M. Holzmann and S. Moroni, Phys. Rev. Lett. 124, 206404 (2020)
[6] M. Holzmann, B. Bernu, and D. M. Ceperley, Phys. Rev. B 74, 104510 (2006)
[7] T. Comparin, R. Bombin, M. Holzmann, F. Mazzanti, J. Boronat, and S. Giorgini, Phys. Rev. A 99, 043609 (2019)

- **Semyon Klevtsov (Strasbourg)**

Coulomb gas vs. Laughlin states on Riemann surfaces

Quantum Hall states, in particular the Laughlin state, is one of the main arenas fro the applications of Coulomb gas. The models are similar on the plane and on the sphere (for integer beta), but start to differ on higher genus surfaces, where Laughlin state is degenerate and Coulomb gas is unique. I will discuss this relation and talk about recent work with Dimitri Zvonkine (Paris-Versaille, CNRS) computing the degeneracy and all Chern classes of the vector bundle of Lauhglin state on the space of Aharonov-Bohm fluxes, establishing Wen-Zee formula and Wen-Niu topological degeneracy.

- **Gaultier Lambert (Zurich)**

Scaling limits of the Gaussian beta-ensemble characteristic polynomial

The Gaussian beta-ensemble or one-dimensional log-gas is a classical model of random matrix theory which describes a gas of electric charges confined on the real line which interact via the two-dimensional Coulomb kernel. I will report on recent asymptotic results for the characteristic polynomial of these ensembles at general inverse-temperature beta. These asymptotics involve the so-called Sine and Airy processes, as well as a Gaussian log-correlated field, and they should be compared to the classical Plancherel-Rotach asymptotics for the Hermite polynomials ($\beta = \infty$). The proof are based on the Dumitriu-Edelman tridiagonal representation of the Gaussian beta-ensemble and the transfer matrix method. If time permits, I will also mention some connections to Gaussian multiplicative chaos. Joint work with Elliot Paquette (McGill University).

- **Thomas Leblé (Paris-Cité)**

About charge fluctuations in the 2D Coulomb gas

I will review some results concerning number statistics for particles in 2D Coulomb gases (and related systems) in the mathematics and the physics literature, emphasizing a few elements that are missing in order to bridge the gap between the two perspectives.

- **Satya Majumdar (Paris-Saclay)**

Harmonically confined Riesz gas in one dimension

I will discuss one dimensional Riesz gas of N particles confined in a harmonic potential. The interaction between any pair of particles at positions x_i and x_j is repulsive and behaves as $\mathrm{sgn}(k)|x_i - x_j|^{-k}$ for $i \neq j$, where $k > -2$. We will first compute the average density in the large N limit explicitly for all $k > -2$. Next, we will compute the exact average density (large N limit) in the presence of a hard wall at $x = w$. Finally, I will discuss the statistics of the position of the rightmost particle in the gas, and will compute the explicit large deviation functions of its distribution. We will see that the left tail exhibits a third order phase transition for all $k > -2$.

• **Rémi Rhodes (Marseille)**

Conformal Bootstrap in Liouville theory

Liouville field theory was introduced by Polyakov in the eighties in the context of string theory. Liouville theory appeared there under the form of a 2D Feynman path integral, describing fluctuating metrics over Riemann surfaces. Since then, this theory has been extensively studied in physics and this interest has more recently spread to the probabilistic community where it appears as a natural model of random Riemann surfaces. Liouville theory is a conformal field theory and, as such, the quantities of interest are the correlation functions. In this talk, we will explain some joint works with G. Baverez, C. Guillarmou and A. Kupiainen where we show that the correlation functions of Liouville conformal field theory on Riemann surfaces can be expressed in terms of products of 3-point correlation functions on the sphere and the conformal blocks, which are holomorphic functions on the moduli space of punctured Riemann surfaces.

• **Nicolas Rougerie (École Normale Supérieure de Lyon)**

On quantum statistics transmutation via flux attachment

We consider a model for two types (bath and tracers) of 2D quantum particles in a perpendicular (artificial) magnetic field. Interactions are short range and only inter-species, and we assume that the bath particles are fermions, all lying in the lowest Landau level of the magnetic field. Heuristic arguments then indicate that, if the tracers are strongly coupled to the bath, they effectively change their quantum statistics, from bosonic to fermionic or vice-versa. We rigorously compute the energy of a natural trial state, indeed exhibiting this phenomenon of statistics transmutation. The proof is based on precise estimates for the characteristic polyno–mial of the Ginibre ensemble of random matrices.

The talk will begin with a review of the relevant quantum mechanical concepts and of the physics heuristics behind the result. I will then explain how we use the Ginibre ensemble as a tool, and how we would very much like to be able to do the same with more general 2D beta-ensembles. This seems very challenging, but would rigorously vindicate the emergence of anyons, i.e. particles whose quantum statistics interpolate between bosonic and fermionic.

Joint work with G. Lambert and D. Lundholm.

• **Benjamin Schlein (Zurich)**

Upper bounds on the ground state energy of dilute Bose gases

We review some recent estimates on the energy of interacting Bose gases in different limits. We conclude presenting a simple new bound for hard spheres in the thermodynamic limit, resolving the ground state energy up to an error comparable with the so-called Lee-Huang-Yang corrections.

• **Grégory Schehr (Paris-Saclay)**

Local statistics in the 1d-Coulomb gas: extreme, gap and full-counting statistics

I will present results for the local fluctuations in the one-dimensional Coulomb gas in the presence of a confining harmonic potential – the so-called "jellium model" – with N particles. I will discuss three distinct observables: the position of the rightmost particle, the gap between the positions of two consecutive particles and the number of particles in a given interval $[-L, L]$ centred around the origin. In all cases, for large N, I will discuss both the typical and the atypical fluctuations, characterized respectively by scaling and large deviation functions, that can be computed explicitly.

- **Robert Seiringer (IST Vienna)**

On the sharp constant in the Lieb–Oxford inequality

The Lieb–Oxford inequality provides a lower bound on the Coulomb energy of a classical system of identical charges in terms of their one-particle density. We show a new estimate on the best constant in this inequality, which, with the help of numerical evaluation, provides the upper bound of 1.58. The best constant has recently been shown to be larger than 1.44, the minimal energy per particle of a uniform system at unit density. (Joint work with Mathieu Lewin and Elliott Lieb.)

- **Jean-Marie Stéphan (Lyon)**

Edge behavior in the 2d and 4d Quantum Hall effect

I discuss two aspects of the edge behavior of quantum Hall droplets, in two and four dimensions. In the first part, I construct explicit Slater determinant wave functions corresponding to particles in a magnetic field trapped by a harmonic potential, generalizing the two-dimensional Integer Quantum Hall effect to four dimensions. I then study the asymptotics of the propagator (or correlation kernel) at the edge, and interpret those in terms of one dimensional edge trajectories. The result depends strongly on the commensurability of the trapping frequencies. In particular incommensurable frequencies produce quasi-periodic, ergodic trajectories; the correlation kernel also exhibits fractal-like features. In the second part, I revisit the problem of the edge density profile in Laughlin states, or equivalently the two dimensional one component coulomb gas with logarithmic interactions. I argue that the decay and period of the (numerically observed) density oscillations can be explained within a rough physical picture of Wigner crystallization near the boundary, where the crystal is gradually melted with the increasing distance to the boundary.

- **Jan Philip Solovej (Copenhagen)**

Universality in the structure of complex atoms

I will discuss the ionization conjecture for atoms (and molecules). It states that the radius, ionization energy, and maximal negative ionization remain bounded as the atomic number becomes large. I will review some known rigorous results, in particular, the Hartree-Fock case. I will compare with experimental and numerical values. I will also discuss what is not known and what might be a picture of an infinite periodic table.

- **Jakob Yngvason (Vienna)**

Quantum Hall physics in higher Landau levels

The quantum states of charged particles moving in a plane orthogonal to a homogeneous magnetic field are naturally grouped into Landau levels which corres–pond to quantization of the cyclotron motion of the particles in the magnetic field. The interplay of this motion with external electric fields and impurities as well as Coulomb interactions between the particles is the subject of quantum Hall physics which has in the past 40 years developed into a major subfield of condensed matter physics.

In the talk I shall review one aspect of quantum Hall physics, namely the relations between wave functions and effective Hamiltonians in different Landau levels. These relations can in particular be used to transport to higher Landau bounds on particle densities rooted in the analyticty of wave functions in the lowest Landau level I shall also briefly comment on states in the second Landau level with filling fraction 5/2 which have recently received much attention from physicists due to potential applications in quantum computing.